

# NONNOETHERIAN HOMOTOPY DIMER ALGEBRAS AND NONCOMMUTATIVE CREPANT RESOLUTIONS

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**ABSTRACT.** Cancellative dimer algebras form a prominent class of examples of noncommutative crepant resolutions (NCCRs). However, dimer algebras which are cancellative are quite rare, and we consider the question: how close are nonnoetherian homotopy dimer algebras to being NCCRs? To address this question, we introduce a generalization of NCCRs to nonnoetherian tiled matrix rings. We show that if a cancellative dimer algebra is obtained from a nonnoetherian homotopy dimer algebra  $A$  by contracting each arrow whose head has indegree 1, then  $A$  is a noncommutative desingularization of its nonnoetherian center. Furthermore, if any two arrows whose tails have indegree 1 are coprime, then  $A$  is a nonnoetherian NCCR.

## 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a local domain with an algebraically closed residue field  $k$ . In the mid 1950's, Auslander, Buchsbaum, and Serre established the famous homological characterization of regularity in the case  $R$  is noetherian [AB, AB2, S]:  $R$  is regular if and only if

$$\mathrm{gldim} R = \mathrm{pd}_R(k) = \dim R.$$

In 1984, Brown and Hajarnavis generalized this characterization to the setting of noncommutative noetherian rings which are module-finite over their centers [BH]: such a ring  $A$  with local center  $R$  is said to be homologically homogeneous if for each simple  $A$ -module  $V$ ,

$$\mathrm{gldim} A = \mathrm{pd}_A(V) = \dim R.$$

In 2002, Van den Bergh placed this notion in the context of derived categories with the introduction of noncommutative crepant resolutions (henceforth NCCRs). Specifically, a homologically homogeneous ring  $A$  is a (local) NCCR if  $R$  is a normal Gorenstein domain and  $A$  is the endomorphism ring of a finitely generated reflexive  $R$ -module [V, Definition 4.1].<sup>1</sup>

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<sup>1</sup>A proper birational map  $f : Y \rightarrow X$  from a non-singular variety  $Y$  to a Gorenstein singularity  $X$  is a crepant resolution if  $f^*\omega_X = \omega_Y$ . Given an NCCR  $A$  of  $R = k[X]$ , Van den Bergh conjectured that the bounded derived category of  $A$ -modules is equivalent to the bounded derived category of coherent sheaves on  $Y$  [V, Conjecture 4.6].

A prominent class of NCCRs are cancellative dimer algebras (Definition 2.2) [Br, Bo, D, B]. In fact, every 3-dimensional affine toric Gorenstein singularity admits an NCCR given by a cancellative dimer algebra [G, IU]. A *homotopy algebra* is the quotient of a dimer algebra by homotopy-like relations on the paths in its quiver; a dimer algebra is then cancellative iff it coincides with its homotopy algebra. Homotopy algebras, just like cancellative dimer algebras, are tiled matrix rings over polynomial rings [B3, Section 4.3]. The homotopy algebra of a non-cancellative dimer algebra is typically nonnoetherian and an infinitely generated module over its nonnoetherian center. Here we consider the question:

*How close are nonnoetherian homotopy algebras to being NCCRs?*

To address this question, we consider a relatively small but important class of nonnoetherian homotopy algebras: Let  $A$  be a homotopy algebra such that a cancellative dimer algebra is obtained by contracting each arrow whose head has indegree 1, and no arrow of  $A$  has head and tail of indegree both 1. Denote by  $R$  the center of  $A$ . The scheme  $\text{Spec } R$  has a unique closed point  $\mathfrak{m}_0$  of positive geometric dimension [B3, Theorem 4.68]. Furthermore,  $\mathfrak{m}_0$  is the unique closed point for which the localizations

$$R_{\mathfrak{m}_0} \quad \text{and} \quad A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$$

are nonnoetherian [B3, Lemma 4.55, Theorem 4.65], [B4, Theorem 3.4]. An initial answer to our question appears to be negative:

- $A_{\mathfrak{m}_0}$  has infinite global dimension (Proposition 6.1).
- $A_{\mathfrak{m}_0}$  is typically not the endomorphism ring of a module over its center.

However, the underlying structure of  $A_{\mathfrak{m}_0}$  is more subtle. To uncover this structure, we introduce a generalization of homological homogeneity and NCCRs for nonnoetherian tiled matrix rings. Let  $A$  be a nonnoetherian tiled matrix ring with local center  $(R, \mathfrak{m})$ . Firstly, we introduce

- the *cycle algebra*  $S$  of  $A$ , which is a commutative algebra that contains the center  $R$  as a subalgebra (but in general is not a subalgebra of  $A$ ); and
- the *cyclic localization*  $A_{\mathfrak{q}}$  of  $A$  at a prime ideal  $\mathfrak{q}$  of  $S$ .

We then say  $A$  is *cycle regular* if for each  $\mathfrak{q} \in \text{Spec } S$  minimal over  $\mathfrak{m}$  and each simple  $A_{\mathfrak{q}}$ -module  $V$ , we have

$$\text{gldim } A_{\mathfrak{q}} = \text{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}}.$$

Furthermore, we say  $A$  is a *nonnoetherian NCCR* if the cycle algebra  $S$  is a noetherian normal Gorenstein domain,  $A$  is cycle regular, and for each  $\mathfrak{q} \in \text{Spec } S$  minimal over  $\mathfrak{m}$ ,  $A_{\mathfrak{q}}$  is the endomorphism ring of a reflexive  $Z(A_{\mathfrak{q}})$ -module.

Our main result is the following.

**Theorem 1.1.** (Theorems 5.7, 6.15, 7.8.) *Let  $A$  be a nonnoetherian homotopy algebra such that a cancellative dimer algebra is obtained by contracting each arrow whose head has indegree 1, and no arrow of  $A$  has head and tail of indegree both 1. Then*

- (1)  $A_{\mathfrak{m}_0}$  is cycle regular.
- (2) For each prime ideal  $\mathfrak{q}$  of the cycle algebra  $S$  which is minimal over  $\mathfrak{m}_0$ ,

$$\mathrm{gldim} A_{\mathfrak{q}} = \dim S_{\mathfrak{q}} = \mathrm{ght}_R(\mathfrak{m}_0) = 1 < 3 = \mathrm{ht}_R(\mathfrak{m}_0) = \dim R_{\mathfrak{m}_0},$$

where  $\mathrm{ght}_R(\mathfrak{m}_0)$  and  $\mathrm{ht}_R(\mathfrak{m}_0)$  denote the geometric height and height of  $\mathfrak{m}_0$  in  $R$  respectively. Furthermore, for each prime  $\mathfrak{q}$  of  $S$  minimal over  $\mathfrak{q} \cap R$ ,

$$\mathrm{gldim} A_{\mathfrak{q}} = \mathrm{ght}_R(\mathfrak{q} \cap R).$$

- (3) If the arrows whose tails have indegree 1 are pair-wise coprime, then  $A_{\mathfrak{m}_0}$  is a nonnoetherian NCCR.

The second claim suggests that geometric height, *not* height, is the ‘right’ notion of codimension for nonnoetherian commutative rings, noting that geometric height and height coincide for noetherian rings [B2, Theorem 3.8]. An example of a dimer algebra which is a nonnoetherian NCCR is given in Figure 1, and described in Section 7.1.

This work is a continuation of [B4], where the author considered localizations  $A_{\mathfrak{p}} := A \otimes_R R_{\mathfrak{p}}$  of nonnoetherian dimer algebras  $A$  at points  $\mathfrak{p} \in \mathrm{Spec} R$  away from  $\mathfrak{m}_0$ . We focus exclusively on homotopy algebras here since the localization of a dimer algebra at  $\mathfrak{m}_0$  is much less tractable than its homotopy counterpart; for example, any dimer algebra satisfying the assumptions of Theorem 1.1 has a free subalgebra, whereas its homotopy algebra does not [B3, Proposition 4.47].

In future work we hope to explore the implications of the definitions we have introduced in terms of derived categories and tilting theory, and to study larger classes of nonnoetherian homotopy algebras, as well as other classes of tiled matrix rings ([B5] is forthcoming).

## 2. PRELIMINARY DEFINITIONS

Throughout, let  $k$  be an algebraically closed field, let  $S$  be an integral domain and a  $k$ -algebra, and let  $R$  be a (possibly nonnoetherian) subalgebra of  $S$ . Denote by  $\mathrm{Max} S$ ,  $\mathrm{Spec} S$ , and  $\dim S$  the maximal spectrum (or variety), prime spectrum (or affine scheme), and Krull dimension of  $S$  respectively; similarly for  $R$ . For a subset  $I \subset S$ , set  $\mathcal{Z}(I) := \{\mathfrak{n} \in \mathrm{Max} S \mid \mathfrak{n} \supseteq I\}$ .

A quiver  $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$  consists of a vertex set  $Q_0$ , an arrow set  $Q_1$ , and head and tail maps  $\mathrm{h}, \mathrm{t} : Q_1 \rightarrow Q_0$ . Denote by  $\deg^+ i$  the indegree of a vertex  $i \in Q_0$ . By module and global dimension we mean left module and left global dimension, unless stated otherwise.

The following definitions were introduced in [B2] to formulate a theory of geometry for nonnoetherian rings with finite Krull dimension.

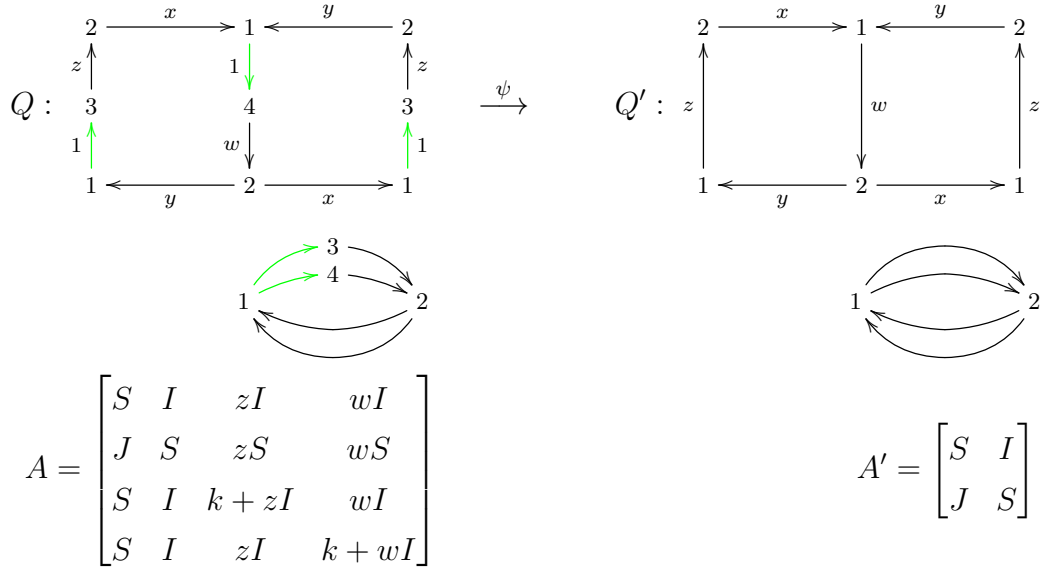


FIGURE 1. The homotopy algebra  $A$  is a nonnoetherian NCCR; see Section 7.1. The quivers  $Q$  and  $Q'$  on the top line are each drawn on a torus, and the two contracted arrows of  $Q$  are drawn in green.

**Definition 2.1.** [B2, Definition 3.1]

- We say  $S$  is a *depiction* of  $R$  if  $S$  is a finitely generated  $k$ -algebra, the morphism

$$\iota_{S/R} : \operatorname{Spec} S \rightarrow \operatorname{Spec} R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is surjective, and

$$\{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}\} = \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n} \cap R} \text{ is noetherian}\} \neq \emptyset.$$

- The *geometric height* of  $\mathfrak{p} \in \operatorname{Spec} R$  is the minimum

$$\operatorname{ght}(\mathfrak{p}) := \min \left\{ \operatorname{ht}_S(\mathfrak{q}) \mid \mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p}), S \text{ a depiction of } R \right\}.$$

The *geometric dimension* of  $\mathfrak{p}$  is

$$\operatorname{gdim} \mathfrak{p} := \dim R - \operatorname{ght}(\mathfrak{p}).$$

The algebras that we will consider here are called homotopy (dimer) algebras. Dimer algebras are a type of quiver with potential, and were introduced in string theory [BFHMS] (see also [BD]). Homotopy algebras are special quotients of dimer algebras, and were introduced in [B3, Definition 4.33].

**Definition 2.2.**

- Let  $Q$  be a finite quiver whose underlying graph  $\overline{Q}$  embeds into a two-dimensional real torus  $T^2$ , such that each connected component of  $T^2 \setminus \overline{Q}$  is simply connected and

bounded by an oriented cycle of length at least 2, called a *unit cycle*.<sup>2,3</sup> The *dimer algebra* of  $Q$  is the quiver algebra  $kQ/I$  with relations

$$I := \langle p - q \mid \exists a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ,$$

where  $p$  and  $q$  are paths.

• Two paths  $p, q \in kQ/I$  form a *non-cancellative pair* if  $p \neq q$ , and there is a path  $r \in kQ/I$  such that

$$rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0.$$

$kQ/I$  and  $Q$  are called *non-cancellative* if there is a non-cancellative pair; otherwise they are called *cancellative*.

• We call the quotient algebra

$$A := (kQ/I) / \langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle$$

the *homotopy (dimer) algebra* of  $Q$ .<sup>4</sup>

- Let  $A$  be a (homotopy) dimer algebra with quiver  $Q$ .
  - A *perfect matching*  $D \subset Q_1$  is a set of arrows such that each unit cycle contains precisely one arrow in  $D$ .
  - A *simple matching*  $D \subset Q_1$  is a perfect matching such that  $Q \setminus D$  supports a simple  $A$ -module of dimension  $1^{Q_0}$  (that is,  $Q \setminus D$  contains a cycle that passes through each vertex of  $Q$ ). Denote by  $\mathcal{S}$  the set of simple matchings of  $A$ .

### 3. CYCLE ALGEBRA AND NONNOETHERIAN NCCRS

In this section we introduce the cycle algebra, cyclic localizations, and nonnoetherian NCCRs. Let  $B$  be an integral domain and a  $k$ -algebra. Let

$$A = [A^{ij}] \subset M_d(B)$$

be a tiled matrix algebra; that is, each diagonal entry  $A^i := A^{ii}$  is a unital subalgebra of  $B$ .

**Definition 3.1.** Set

$$R := k \left[ \bigcap_{i=1}^d A^i \right] \quad \text{and} \quad S := k \left[ \bigcup_{i=1}^d A^i \right].$$

We call  $S$  the *cycle algebra* of  $A$ . Furthermore, for  $\mathfrak{q} \in \text{Spec } S$ , set

$$A_{\mathfrak{q}} := \left\langle \begin{bmatrix} A_{\mathfrak{q} \cap A^1}^1 & A^{12} & \cdots & A^{1d} \\ A^{21} & A_{\mathfrak{q} \cap A^2}^2 & & \\ \vdots & & \ddots & \\ A^{d1} & & & A_{\mathfrak{q} \cap A^d}^d \end{bmatrix} \right\rangle \subset M_d(\text{Frac } B).$$

<sup>2</sup>In contexts such as cluster algebras,  $\bar{Q}$  may be embedded into any compact surface; see for example [BKM].

<sup>3</sup>Note that for any vertex  $i \in Q_0$ , the indegree and outdegree of  $i$  are equal.

<sup>4</sup>Note that a dimer algebra coincides with its homotopy algebra if and only if its quiver is cancellative.

We call  $A_{\mathfrak{q}}$  the *cyclic localization* of  $A$  at  $\mathfrak{q}$ .

Note that  $R$  and  $S$  are integral domains since they are subalgebras of  $B$ . The following definitions aim to generalize homological homogeneity and NCCRs to the nonnoetherian setting.

**Definition 3.2.** Suppose  $R$  is a local domain with unique maximal ideal  $\mathfrak{m}$ .

- We say  $A$  is *cycle regular* if for each  $\mathfrak{q} \in \operatorname{Spec} S$  minimal over  $\mathfrak{m}$  and each simple  $A_{\mathfrak{q}}$ -module  $V$ ,

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}}.$$

- We say  $A$  is a *noncommutative desingularization* if  $A$  is cycle regular, and  $A \otimes_R \operatorname{Frac} R$  and  $\operatorname{Frac} R$  are Morita equivalent.
- We say  $A$  is a *nonnoetherian noncommutative crepant resolution* if  $S$  is a normal Gorenstein domain,  $A$  is cycle regular, and for each  $\mathfrak{q} \in \operatorname{Spec} S$  minimal over  $\mathfrak{m}$ ,  $A_{\mathfrak{q}}$  is the endomorphism ring of a reflexive  $Z(A_{\mathfrak{q}})$ -module.

**Remark 3.3.** Suppose  $B$  is a finitely generated  $k$ -algebra, and  $k$  is uncountable. Further suppose the embedding  $\tau : A \hookrightarrow M_d(B)$  has the properties that

- (i) for generic  $\mathfrak{b} \in \operatorname{Max} B$ , the composition

$$A \xrightarrow{\tau} M_d(B) \xrightarrow{1} M_d(B/\mathfrak{b})$$

is surjective;

- (ii) the morphism

$$\operatorname{Max} B \rightarrow \operatorname{Max} \tau(Z), \quad \mathfrak{b} \mapsto \mathfrak{b}1_d \cap \tau(Z),$$

is surjective; and

- (iii) for each  $\mathfrak{n} \in \operatorname{Max} S$ ,  $R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}$  iff  $R_{\mathfrak{n} \cap R}$  is noetherian.

$(\tau, B)$  is then said to be an *impression* of  $A$  [B, Definition 2.1].

With these conditions, the center  $Z$  of  $A$  is equal to  $R$ ,

$$Z = R1_d,$$

and is depicted by  $S$  [B2, Theorem 4.1.1]. Furthermore, by [B2, Theorem 4.1.2],

$$\begin{aligned} R = S &\Leftrightarrow A \text{ is a finitely generated } R\text{-module} \\ &\Leftrightarrow R \text{ is noetherian} \\ &\Rightarrow A \text{ is noetherian} \end{aligned}$$

In particular, if  $R$  is noetherian, then the cyclic and central localizations of  $A$  at  $\mathfrak{q} \in \operatorname{Spec} S$  are isomorphic algebras,

$$A_{\mathfrak{q}} \cong A \otimes_R R_{\mathfrak{q} \cap R}.$$

If  $\mathfrak{p} \in \operatorname{Spec} R$  and  $\mathfrak{q} \in \operatorname{Spec} S$ , then we denote by  $A_{\mathfrak{p}}$  and  $A_{\mathfrak{q}}$  the central and cyclic localizations of  $A$  respectively; no ambiguity arises since the two localizations coincide whenever  $R = S$ .

## 4. A CLASS OF NONNOETHERIAN HOMOTOPY ALGEBRAS

For the remainder of this article, we will consider a class of homotopy algebras whose quivers contain vertices with indegree 1. Such quivers are necessarily non-cancellative. Unless stated otherwise, let  $A$  be a nonnoetherian homotopy algebra with quiver  $Q = (Q_0, Q_1, t, h)$  such that

- (A) a cancellative dimer algebra  $A' = kQ'/I'$  is obtained by contracting each arrow of  $Q$  whose head has indegree 1; and
- (B) for each  $a \in Q_1$ , the indegrees  $\deg^+ t(a)$  and  $\deg^+ h(a)$  are not both 1.

Set

$$Q_1^* = \{a \in Q_1 \mid \deg^+ h(a) = 1\} \quad \text{and} \quad Q_1^t := \{a \in Q_1 \mid \deg^+ t(a) = 1\}.$$

The quiver  $Q' = (Q'_0, Q'_1, t', h')$  is then defined by

$$Q'_0 = Q_0 / \{h(a) \sim t(a) \mid a \in Q_1^*\}, \quad Q'_1 = Q_1 \setminus Q_1^*,$$

and for each arrow  $a \in Q'_1$ ,

$$h'(a) = h(a) \quad \text{and} \quad t'(a) = t(a).$$

The homotopy algebras  $A$  and  $A'$  are isomorphic to tiled matrix rings. Indeed, consider the  $k$ -linear map

$$\psi : A \rightarrow A'$$

defined by

$$\psi(a) = \begin{cases} a & \text{if } a \in Q_0 \cup Q_1 \setminus Q_1^* \\ e_{t(a)} & \text{if } a \in Q_1^* \end{cases}$$

and extended multiplicatively to (nonzero) paths and  $k$ -linearly to  $A$ . Furthermore, consider the polynomial ring generated by the simple matchings of  $A'$ ,

$$B = k[x_D \mid D \in \mathcal{S}'].$$

Denote by  $E_{ij}$  the matrix with a 1 in the  $ij$ -th slot and zeros elsewhere.  $A'$  and  $A$  then admit algebra monomorphisms [B3, Theorem 4.35]

$$\tau : A' \hookrightarrow M_{|Q'_0|}(B) \quad \text{and} \quad \tau_\psi : A \hookrightarrow M_{|Q_0|}(B)$$

defined by

$$\tau(a) = \begin{cases} E_{ii} & \text{if } a = e_i \in Q'_0 \\ \left(\prod_{D \in \mathcal{S}' : D \ni a} x_D\right) E_{h(a), t(a)} & \text{if } a \in Q'_1 \end{cases}$$

$$\tau_\psi(a) = \begin{cases} E_{ii} & \text{if } a = e_i \in Q_0 \\ \left(\prod_{D \in \mathcal{S}' : D \ni \psi(a)} x_D\right) E_{h(a), t(a)} & \text{if } a \in Q_1 \end{cases}$$

and extended multiplicatively and  $k$ -linearly to  $A'$  and  $A$ .

For  $p \in e_j A e_i$  and  $p' \in e_j A' e_i$ , denote by

$$\bar{\tau}_\psi(p) = \bar{p} \in B \quad \text{and} \quad \bar{\tau}(p') = \bar{p}' \in B$$

the single nonzero matrix entry of  $\tau_\psi(p)$  and  $\tau(p')$ , respectively. Note that

$$\bar{\tau}_\psi(p) = \bar{\tau}(\psi(p)).$$

Furthermore, for each  $a \in Q_1$  and  $D \in \mathcal{S}'$ ,

$$x_D | \bar{a} \quad \text{if and only if} \quad \psi(a) \in D.$$

**Lemma 4.1.**

(1) *The cycle algebras of  $A$  and  $A'$  are equal,*<sup>5</sup>

$$S := k[\cup_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i)] = k[\cup_{i \in Q'_0} \bar{\tau}(e_i A' e_i)].$$

(2) *The center  $Z'$  of  $A'$  is isomorphic to  $S$ , and the center  $Z$  of  $A$  is isomorphic to the intersection*

$$Z \cong R := k[\cap_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i)].$$

(3)  *$S$  is a depiction of  $R$ .*

(4) *If the indegree of a vertex  $i \in Q_0$  is at least 2, then*

$$\bar{\tau}_\psi(e_i A e_i) = S.$$

*In particular, for each arrow  $a \in Q_1$ ,*

$$\bar{\tau}_\psi(e_{t(a)} A e_{t(a)}) = S \quad \text{or} \quad \bar{\tau}_\psi(e_{h(a)} A e_{h(a)}) = S.$$

*Proof.* (1) By assumption (A), for each cycle  $p'$  in  $Q'$ , there is a cycle  $p$  in  $Q$  such that  $\psi(p) = p'$ . Therefore the cycle algebras of  $A$  and  $A'$  are equal.

(2) Since  $A'$  is cancellative, for each  $i, j \in Q'_0$ ,

$$\bar{\tau}(e_i A' e_i) = \bar{\tau}(e_j A' e_j),$$

by [B3, Theorem 4.35]. Whence for each  $i \in Q'_0$ ,

$$(1) \quad \bar{\tau}(e_i A' e_i) = S.$$

Furthermore, the centers  $Z$  and  $Z'$  are isomorphic to the intersections

$$Z \cong R := k[\cap_{i \in Q_0} \bar{\tau}_\psi(e_i A e_i)] \quad \text{and} \quad Z' \cong k[\cap_{i \in Q'_0} \bar{\tau}(e_i A' e_i)],$$

by [B3, Theorems 3.5 and 4.35]. Therefore  $Z'$  is isomorphic to  $S$  by (1).

(3) Since  $A$  and  $A'$  have equal cycle algebras,  $Z \cong R$  is depicted by  $Z' \cong S$ , by [B3, Theorem 4.68.1].

(4) By assumption (A), if a vertex  $i \in Q_0$  has indegree at least 2, then

$$\bar{\tau}_\psi(e_i A e_i) = \bar{\tau}(e_{\psi(i)} A' e_{\psi(i)}) \stackrel{(1)}{=} S,$$

where (1) holds by (1). Furthermore, by assumption (B), the head or tail of each arrow  $a \in Q_1$  has indegree at least 2.  $\square$

<sup>5</sup>The map  $\psi$  is therefore a ‘cyclic contraction’ [B3, Definitions 4.1 and 4.3].



## 5. PRIME DECOMPOSITION OF THE ORIGIN

Consider the origin of  $\text{Max } R$ ,

$$\mathfrak{m}_0 := (x_D \mid D \in \mathcal{S}') B \cap R.$$

For a monomial  $g \in B$ , denote by  $\mathfrak{q}_g$  the ideal in  $S$  generated by all monomials in  $S$  which are divisible by  $g$  in  $B$ . If  $g = x_D$  for some simple matching  $D \in \mathcal{S}'$ , then set

$$\mathfrak{q}_D := \mathfrak{q}_{x_D}.$$

We will write  $h \mid g$  if  $h$  divides  $g$  in  $B$ , unless stated otherwise.

**Lemma 5.1.** *Let  $g \in B$  be a monomial. Then the ideal  $\mathfrak{q}_g \subset S$  is prime if and only if  $g = x_D$  for some  $D \in \mathcal{S}'$ .*

*Proof.* Enumerate the simple matchings of  $A'$ ,  $\mathcal{S}' = \{D_1, \dots, D_n\}$ , and set  $x_i := x_{D_i}$ .

(i) We first claim that for each pair of distinct simple matchings  $D_i, D_j \in \mathcal{S}'$ , there is a cycle  $s \in A$  satisfying

$$(2) \quad x_i \mid \bar{s} \quad \text{and} \quad x_j \nmid \bar{s}.$$

Indeed, fix  $i \neq j$ . Since  $D_i \neq D_j$ , there is an arrow  $a \in Q'_1$  for which  $a \in D_i \setminus D_j$ . Furthermore, since  $D_j$  is simple, there is a path  $p \in e_{t(a)} A' e_{h(a)}$  supported on  $Q' \setminus D_j$ . Whence  $s := pa$  is a cycle satisfying (2). But  $A$  and  $A'$  have equal cycle algebras by Lemma 4.1.1. Therefore  $\bar{s}$  is the  $\bar{\tau}_\psi$ -image of a cycle in  $A$ , proving our claim.

(ii) We now claim that if  $g \in B$  is a monomial and  $\mathfrak{q}_g$  is a prime ideal of  $S$ , then  $g = x_D$  for some  $D \in \mathcal{S}'$ . It suffices to consider a monomial  $g = \prod_{i=1}^{n'} x_i^{m_i}$ , where  $2 \leq n' \leq n$ , and for each  $i$ ,  $m_i \geq 1$ . By Claim (i), there are cycles  $s_1, \dots, s_{n'} \in A$  such that

$$x_1 \mid \bar{s}_1, \quad x_2 \nmid \bar{s}_1,$$

and for each  $2 \leq i \leq n'$ ,

$$x_1 \nmid \bar{s}_i, \quad x_i \mid \bar{s}_i.$$

Set

$$h_1 := \bar{s}_1^{m_1} \quad \text{and} \quad h_2 := \prod_{i=2}^{n'} \bar{s}_i^{m_i}.$$

Then  $h_1 h_2 \in \mathfrak{q}_g$ . But  $h_1 \notin \mathfrak{q}_g$  and  $h_2 \notin \mathfrak{q}_g$  since  $x_2 \nmid h_1$  and  $x_1 \nmid h_2$ . Therefore  $\mathfrak{q}_g$  is not prime.

(iii) Finally, consider a simple matching  $D \in \mathcal{S}'$ . If  $s, t \in e_i A e_i$  are cycles for which  $x_D \mid \bar{s} \bar{t}$ , then  $x_D \mid \bar{s}$  or  $x_D \mid \bar{t}$ , since  $B$  is the polynomial ring generated by  $\mathcal{S}'$ . Therefore the ideal  $\mathfrak{q}_{x_D}$  is prime.  $\square$

**Lemma 5.2.** *Let  $i, j \in Q_0$  and  $D \in \mathcal{S}'$ . If  $i$  is not the tail of an arrow  $a \in Q_1^t$  for which  $x_D \mid \bar{a}$ , then there is a path  $p \in e_j A e_i$  such that  $x_D \nmid \bar{p}$ .*

*Proof.* (i) First suppose  $\deg^+ i \geq 2$ . Since  $D$  is simple, there is a path  $q \in e_{\psi(j)}A'e_{\psi(i)}$  supported on  $Q' \setminus D$ ; whence  $x_D \nmid \bar{q}$ . Furthermore, since  $\deg^+ i \geq 2$ , there is a path  $p \in e_jAe_i$  such that  $\psi(p) = q$ , by assumption (A). In particular,  $x_D \nmid \bar{q} = \bar{p}$ .

(ii) Now suppose  $\deg^+ i = 1$ . Let  $a \in Q_1^t$  be such that  $t(a) = i$ . Then  $\deg^+ h(a) \geq 2$  by assumption (B). Thus there is a path  $t \in e_jAe_{h(a)}$  for which  $x_D \nmid \bar{t}$ , by Claim (i). Therefore if  $x_D \nmid \bar{a}$ , then the path  $p := ta \in e_jAe_i$  satisfies  $x_D \nmid \bar{p}$ .  $\square$

**Notation 5.3.** Denote by  $\sigma_i$  the unit cycle at vertex  $i \in Q_0$ , and by

$$\sigma := \bar{\tau}_\psi(\sigma_i) = \prod_{D \in S'} x_D$$

the common  $\bar{\tau}_\psi$ -image of each unit cycle in  $Q$ . ( $\sigma$  is also the  $\bar{\tau}$ -image of each unit cycle in  $Q'$ .) Furthermore, consider a covering map of the torus,  $\pi : \mathbb{R}^2 \rightarrow T^2$ , such that for some  $i \in Q_0$ ,

$$\pi(\mathbb{Z}^2) = i.$$

Denote by

$$Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$$

the covering quiver of  $Q$ . For each path  $p$  in  $Q$ , denote by  $p^+$  a path in  $Q^+$  with tail in  $[0, 1) \times [0, 1) \subset \mathbb{R}^2$  satisfying  $\pi(p^+) = p$ .

**Lemma 5.4.** *Let  $a \in A'$  be an arrow and let  $s \in e_{t(a)}A'e_{t(a)}$  be a cycle satisfying  $\bar{a} \mid \bar{s}$ . Then there is a path  $p \in e_{t(a)}A'e_{h(a)}$  such that*

$$s = pa.$$

*Proof.* We use the notation in [B4, Notation 2.1]. Suppose the hypotheses hold.<sup>6</sup> It suffices to assume  $\sigma \nmid \bar{s}$  by [B3, Lemma 1.5]. Whence  $s \in \hat{\mathcal{C}}$  by [B3, Lemma 2.8.3]. Let  $u \in \mathbb{Z}^2$  be such that  $s \in \hat{\mathcal{C}}^u$ . Since  $A'$  is cancellative, for each  $i \in Q'_0$  we have

$$(3) \quad \hat{\mathcal{C}}_i^u \neq \emptyset,$$

by [B3, Proposition 2.10]. Consider  $t \in \hat{\mathcal{C}}_{h(a)}^u$ . Then  $\bar{s} = \bar{t}$  by [B3, Proposition 2.20.2].

Now the paths  $(as)^+$  and  $(ta)^+$  bound a compact region

$$\mathcal{R}_{as,ta} \subset \mathbb{R}^2.$$

Furthermore, since  $A'$  is cancellative, if a cycle  $p$  is formed from subpaths of cycles in  $\hat{\mathcal{C}}^u$ , then  $p$  is in  $\hat{\mathcal{C}}^u$ , by [B3, Proposition 2.20.3]. Therefore we may suppose that the interior of  $\mathcal{R}_{as,ta}$  does not contain any vertices of  $Q'^+$ , by (3).

First suppose  $s^+$  and  $t^+$  do not intersect (modulo  $I$ ). Then  $a$  is contained in a simple matching  $D$  of  $A'$  such that  $x_D \nmid \bar{s}$ , by [B3, Lemma 2.15]; see Figure 2.i. But  $x_D \mid \bar{s}$  since  $\bar{a} \mid \bar{s}$ , a contradiction.

<sup>6</sup>This proof is similar to [B4, Claim (i) in proof of Lemma 2.4].

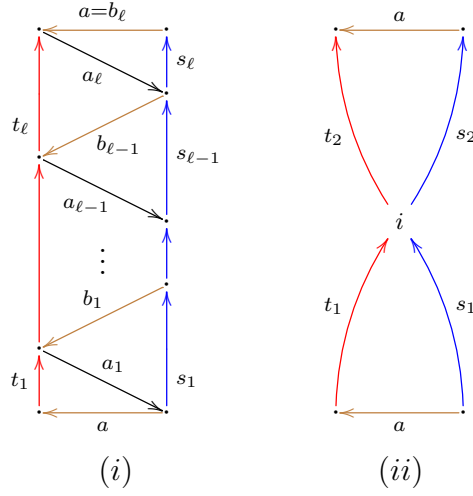


FIGURE 2. Cases for Lemma 5.4. In case (i),  $s$  and  $t$  factor into paths  $s = s_\ell \cdots s_2 s_1$  and  $t = t_\ell \cdots t_2 t_1$ , where  $a_1, \dots, a_\ell, b_1, \dots, b_\ell$  are arrows, and the cycles  $a_j b_j s_j$  and  $b_{j-1} a_j t_j$  are unit cycles. The  $b_j$  arrows, drawn in brown, belong to a simple matching  $D$  of  $A'$ . In case (ii),  $s$  and  $t$  factor into paths  $s = s_2 e_i s_1$  and  $t = t_2 e_i t_1$ .

Therefore  $s^+$  and  $t^+$  intersect at a vertex  $i^+$ ; see Figure 2.ii. By assumption,  $\sigma \nmid \bar{s} = \bar{t}$ . Whence  $\sigma \nmid \bar{a}\bar{s}$  and  $\sigma \nmid \bar{t}\bar{a}$  since  $\bar{a} \mid \bar{s} = \bar{t}$ . Thus

$$\bar{s}_1 = \bar{t}_1 \bar{a} \quad \text{and} \quad \bar{a} \bar{s}_2 = \bar{t}_2,$$

by [B3, Lemma 2.3.2]. Consequently,

$$\overline{s_2 t_1 a} = \bar{s}_2 \bar{s}_1 = \bar{s}.$$

Therefore, since  $\tau : A' \rightarrow M_{|Q'_0|}(B)$  is injective,

$$s_2 t_1 a = s.$$

In particular, we may take  $p = s_2 t_1$ . □

**Proposition 5.5.** *For each arrow  $a \in Q_1 \setminus Q_1^*$ ,  $\bar{\tau}_\psi(e_{t(a)} Aa)$  is an ideal of  $S$  with prime decomposition*

$$(4) \quad \bar{\tau}_\psi(e_{t(a)} Aa) = \bigcap_{D \in S' : x_D \mid \bar{a}} \mathfrak{q}_D.$$

Consequently, the prime decomposition of  $\mathfrak{m}_0 \in \text{Max } R$ , as an ideal of  $S$ , is

$$\mathfrak{m}_0 = \bigcap_{a \in Q_1^t} \bar{\tau}_\psi(e_{t(a)} Aa) = \bigcap_{\substack{D \in S' : \\ x_D \mid \bar{a} \text{ where } a \in Q_1^t}} \mathfrak{q}_D.$$

*Proof.*  $\bar{\tau}_\psi(e_{t(a)}Aa)$  is an ideal of  $S$  by Lemma 4.1.4. Set  $\mathfrak{q}_a := \bigcap_{D \in \mathcal{S}' : x_D \mid \bar{a}} \mathfrak{q}_D$ . The inclusion  $\bar{\tau}_\psi(e_{t(a)}Aa) \subseteq \mathfrak{q}_a$  is clear. So suppose  $t \in e_j A e_j$  is a cycle such that  $\bar{t} \in \mathfrak{q}_a$ , that is,  $\bar{a} \mid \bar{t}$ .

First suppose  $\deg^+ t(a) \geq 2$ . Then  $e_{t(a)} A e_{t(a)} = S e_{t(a)}$  by Lemma 4.1.4. In particular, there is a cycle  $s \in e_{t(a)} A e_{t(a)}$  for which  $\bar{s} = \bar{t}$ . Furthermore, there is a path  $p \in e_{t(a)} A e_{h(a)}$  such that  $s = pa$ , by Lemma 5.4 and assumption (A).

Now suppose  $\deg^+ t(a) = 1$ . Then  $\deg^+ h(a) \geq 2$  by assumption (B). Whence  $e_{h(a)} A e_{h(a)} = S e_{h(a)}$ . In particular, there is a cycle  $s \in e_{h(a)} A e_{h(a)}$  for which  $\bar{s} = \bar{t}$ . Furthermore, there is a path  $p \in e_{t(a)} A e_{h(a)}$  such that  $s = ap$ , again by Lemma 5.4 and assumption (A).

Thus, in either case,

$$\bar{t} = \bar{s} \in \bar{\tau}_\psi(e_{t(a)}Aa).$$

Therefore (4) holds. Finally, each  $\mathfrak{q}_D$  is prime by Lemma 5.1.  $\square$

In the following, we show that although the ideal  $\mathfrak{q}_D$  may not be principal in  $S$ , it becomes principal over the localization  $S_{\mathfrak{q}_D}$ .

**Proposition 5.6.** *Let  $D \in \mathcal{S}'$  and set  $\mathfrak{q} := \mathfrak{q}_D$ . Then the maximal ideal  $\mathfrak{q}S_{\mathfrak{q}}$  of  $S_{\mathfrak{q}}$  is generated by  $\sigma$ ,*

$$\mathfrak{q}S_{\mathfrak{q}} = \sigma S_{\mathfrak{q}}.$$

*Proof.* Let  $g \in \mathfrak{q}$  be a nonzero monomial. Then there is a cycle  $s \in A$  with  $\bar{s} = g$ . By possibly cyclically permuting the arrow subpaths of  $s$ , we may assume  $s$  factors into paths  $s = pa$ , where  $x_D \mid \bar{a}$  and either

- $a \in Q_1 \setminus (Q_1^* \cup Q_1^t)$ , or
- $a = a'\delta$  where  $\delta \in Q_1^*$  and  $a' \in Q_1^t$ .

In either case,  $\deg^+ t(a) \geq 2$ .

Let  $b$  be a path such that  $ba$  is a unit cycle. Then  $x_D \nmid \bar{b}$  since  $x_D \mid \bar{a}$  and  $\bar{ba} = \sigma$ . Furthermore, since  $\deg^+ h(b) = \deg^+ t(a) \geq 2$ , there is a path  $t \in e_{t(b)} A e_{h(b)}$  for which  $x_D \nmid \bar{t}$ , by Lemma 5.2. In particular,  $tp$  and  $tb$  are cycles, and  $x_D \nmid \bar{tb}$ . Whence

$$\bar{tp} \in S \quad \text{and} \quad \bar{tb} \in S \setminus \mathfrak{q}.$$

Therefore

$$g = \bar{a}\bar{p} \frac{\bar{tb}}{\bar{tb}} = \bar{a}\bar{b} \frac{\bar{tp}}{\bar{tb}} = \sigma \frac{\bar{tp}}{\bar{tb}} \in \sigma S_{\mathfrak{q}}.$$

$\square$

Recall that an ideal  $I$  is unmixed if for each minimal prime  $\mathfrak{q}$  over  $I$ ,  $\text{ht}(\mathfrak{q}) = \text{ht}(I)$ .

**Theorem 5.7.**

- (1) *For each  $D \in \mathcal{S}'$ , the height of  $\mathfrak{q}_D$  in  $S$  is 1.*
- (2) *The set of minimal primes of  $S$  over  $\mathfrak{m}_0$  are the ideals  $\mathfrak{q}_D \in \text{Spec } S$  for which  $D$  contains the  $\psi$ -image of some  $a \in Q_1^t$ .*

(3)  $\mathfrak{m}_0$  is an unmixed ideal of  $S$ . Furthermore,  $\mathfrak{m}_0$  has height 1 as an ideal of  $S$  and height 3 as an ideal of  $R$ ,

$$\mathrm{ht}_S(\mathfrak{m}_0) = 1 \quad \text{and} \quad \mathrm{ht}_R(\mathfrak{m}_0) = 3.$$

*Proof.* (1) Set  $\mathfrak{q} := \mathfrak{q}_D$ . Then

$$1 \stackrel{(i)}{\leq} \mathrm{ht}_S(\mathfrak{q}) = \mathrm{ht}_{S_{\mathfrak{q}}}(\mathfrak{q}S_{\mathfrak{q}}) \stackrel{(ii)}{=} \mathrm{ht}_{S_{\mathfrak{q}}}(\sigma S_{\mathfrak{q}}) \stackrel{(iii)}{\leq} 1.$$

Indeed, (i) holds since  $S$  is an integral domain and  $\mathfrak{q}$  is nonzero; (ii) holds by Proposition 5.6; and (iii) holds by Krull's principal ideal theorem.

(2) Follows from Claim (1) and Proposition 5.5.

(3)  $\mathfrak{m}_0$  is a height 1 unmixed ideal of  $S$  by Claims (1) and (2), and Proposition 5.5. Furthermore,  $R$  admits a depiction by Lemma 4.1.3. Thus the height of each maximal ideal of  $R$  equals the Krull dimension of  $R$  by [B2, Lemma 3.7.2]. But the Krull dimension of  $R$  is 3 by [B3, Theorem 4.66]. Therefore  $\mathrm{ht}_R(\mathfrak{m}_0) = 3$ .  $\square$

**Question 5.8.** Let  $K$  be the function field of an algebraic variety. As shown in Theorem 5.7.3, a subset  $\mathfrak{p}$  of  $K$  may be an ideal in different subalgebras of  $K$ , and the height of  $\mathfrak{p}$  depends on the choice of such subalgebra. Is the geometric height of  $\mathfrak{p}$  independent of the choice of subalgebra for which  $\mathfrak{p}$  is an ideal? If this is the case, then the geometric height would be an intrinsic property of an ideal, whereas its height would not be.

The center and cycle algebra of  $A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$  are respectively

$$Z(A_{\mathfrak{m}_0}) \cong R \otimes_R R_{\mathfrak{m}_0} \cong R_{\mathfrak{m}_0} \quad \text{and} \quad S \otimes_R R_{\mathfrak{m}_0} \cong SR_{\mathfrak{m}_0}.$$

**Proposition 5.9.** *The cycle algebra  $SR_{\mathfrak{m}_0}$  of  $A_{\mathfrak{m}_0}$  is a normal Gorenstein domain.*

*Proof.* Let  $\mathfrak{t} \in \mathrm{Spec}(SR_{\mathfrak{m}_0})$  and set  $\mathfrak{q} := \mathfrak{t} \cap S$ .

(i) We claim that

$$(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = S_{\mathfrak{q}}.$$

Clearly  $(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = S_{\mathfrak{q}}R_{\mathfrak{m}_0}$ .<sup>7</sup> It thus suffices to show that

$$(5) \quad S_{\mathfrak{q}}R_{\mathfrak{m}_0} = S_{\mathfrak{q}}.$$

Indeed, we have

$$(6) \quad \mathfrak{t} \cap R \subseteq \mathfrak{m}_0.$$

<sup>7</sup>To show this, note that the elements of  $SR_{\mathfrak{m}_0}$  are of the form  $s/r$ , with  $s \in S$  and  $r \in R \setminus \mathfrak{m}_0$ . Thus an element of  $(SR_{\mathfrak{m}_0})_{\mathfrak{t}}$  is of the form  $\frac{s_1}{r_1}(\frac{s_2}{r_2})^{-1}$ , with  $s_1, s_2 \in S$ ,  $r_1, r_2 \in R \setminus \mathfrak{m}_0$ , and  $\frac{s_2}{r_2} \notin \mathfrak{t}$ . Furthermore,  $\frac{s_2}{r_2} \notin \mathfrak{t}$  and (6) together imply  $s_2 \notin \mathfrak{t}$ . Whence

$$s_2 \in S \setminus (\mathfrak{t} \cap S) = S \setminus \mathfrak{q}.$$

Therefore

$$\frac{s_1}{r_1} \left( \frac{s_2}{r_2} \right)^{-1} = \frac{s_1 r_2}{s_2} \cdot \frac{1}{r_1} \in S_{\mathfrak{q}}R_{\mathfrak{m}_0}.$$

Thus if  $\mathfrak{m}_0 \subseteq \mathfrak{q}$ , then  $\mathfrak{q} \cap R = \mathfrak{m}_0$ . Whence  $R_{\mathfrak{m}_0} \subseteq S_{\mathfrak{q}}$ . In particular,  $S_{\mathfrak{q}}R_{\mathfrak{m}_0} = S_{\mathfrak{q}}$ . Otherwise  $\mathfrak{q} = 0 \subset \mathfrak{m}_0$  by Theorem 5.7.3; whence

$$S_{\mathfrak{q}}R_{\mathfrak{m}_0} = (\text{Frac } S)R_{\mathfrak{m}_0} = \text{Frac } S = S_{\mathfrak{q}}.$$

Therefore in either case (5) holds, proving our claim.

(ii)  $S$  is isomorphic to the center of  $A'$  by Lemma 4.1.2. Thus  $S$  is a normal Gorenstein domain since  $A'$  is an NCCR. Whence  $S_{\mathfrak{q}}$  is a normal Gorenstein domain. But  $(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = S_{\mathfrak{q}}$  by Claim (i). Therefore  $(SR_{\mathfrak{m}_0})_{\mathfrak{t}}$  is a normal Gorenstein domain. Since this holds for all  $\mathfrak{t} \in \text{Spec}(SR_{\mathfrak{m}_0})$ ,  $SR_{\mathfrak{m}_0}$  is also a normal Gorenstein domain.  $\square$

## 6. CYCLE REGULARITY

Unless stated otherwise, let  $\mathfrak{q} \in \text{Spec } S$  be a minimal prime over the origin  $\mathfrak{m}_0$  of  $\text{Max } R$ . In particular, there is a simple matching  $D \in \mathcal{S}'$  such that  $\mathfrak{q} = \mathfrak{q}_D$ , by Proposition 5.5.

We begin by showing that a notion of homological regularity cannot be obtained by considering the central localization  $A_{\mathfrak{m}_0}$  alone.

**Proposition 6.1.** *The  $A_{\mathfrak{m}_0}$ -module  $A_{\mathfrak{m}_0}/\mathfrak{m}_0 = A \otimes_R (R_{\mathfrak{m}_0}/\mathfrak{m}_0)$  has infinite projective dimension, and therefore  $A_{\mathfrak{m}_0}$  has infinite global dimension.*

*Proof.* By [B3, Lemma 4.44], there are monomials  $g, h \in S$  such that for each  $n \geq 1$ ,

$$h^n \notin R \quad \text{and} \quad gh^n \in \mathfrak{m}_0 \subset R.$$

In particular, there is a vertex  $i \in Q_0$  such that for each  $n \geq 1$ ,

$$h^n \notin \bar{\tau}_{\psi}(e_i A e_i).$$

Let  $s_n$  be the cycle in  $e_i A e_i$  satisfying  $\bar{s}_n = gh^n$ . Consider a projective resolution of  $A_{\mathfrak{m}_0}/\mathfrak{m}_0$  over  $A_{\mathfrak{m}_0}$ ,

$$\cdots \rightarrow P_1 \longrightarrow A_{\mathfrak{m}_0} \xrightarrow{\cdot 1} A_{\mathfrak{m}_0}/\mathfrak{m}_0 \rightarrow 0.$$

Each  $s_n$  is in the zeroth syzygy module  $\ker(\cdot 1) = \text{ann}_{A_{\mathfrak{m}_0}}(A_{\mathfrak{m}_0}/\mathfrak{m}_0)$ . Thus  $\ker(\cdot 1)$  is not finitely generated over  $A_{\mathfrak{m}_0}$  since  $h^n \notin \bar{\tau}_{\psi}(e_i A e_i)$ . Furthermore, the cycles  $s_n$  are pair-wise commuting, and in particular there are an infinite number of independent commutation relations between them. It follows that  $\text{pd}_{A_{\mathfrak{m}_0}}(A_{\mathfrak{m}_0}/\mathfrak{m}_0) = \infty$ .  $\square$

**Lemma 6.2.** *Let  $V$  be a simple  $A_{\mathfrak{q}}$ -module, and let  $i \in Q_0$ . Then*

$$\dim_k e_i V \leq 1.$$

*Proof.* Suppose  $V$  is a simple  $A_{\mathfrak{q}}$ -module. Then  $e_i V$  is a simple  $e_i A_{\mathfrak{q}} e_i$ -module. Furthermore, the corner ring  $e_i A_{\mathfrak{q}} e_i \cong \bar{\tau}_{\psi}(e_i A_{\mathfrak{q}} e_i) \subset B$  is a commutative  $k$ -algebra and  $k$  is algebraically closed. Therefore  $\dim_k e_i V \leq 1$  by Schur's lemma.  $\square$

**Lemma 6.3.** *Let  $V$  be a simple  $A_{\mathfrak{q}}$ -module, and let  $i \in Q_0$  be a vertex for which  $e_i V \neq 0$ . Suppose  $s \in e_i A_{\mathfrak{q}} e_i$ . Then  $sV = 0$  if and only if  $\bar{s} \in \mathfrak{q}$ . Consequently,  $\text{ann}_R V = \mathfrak{m}_0$ .*

*Proof.* (i) First suppose  $s \in e_i A e_i$  satisfies  $\bar{s} \in \mathfrak{q}$ . We claim that  $sV = 0$ .

Indeed, let  $v \in e_i V$  be nonzero. Then  $\dim_k e_i V = 1$  by Lemma 6.2. Thus there is some  $c \in k$  such that  $(s - ce_i)e_i V = 0$ . Assume to the contrary that  $c$  is nonzero. Then  $\bar{s} - c \in S \setminus \mathfrak{q}$ . Therefore

$$v = \frac{s - ce_i}{\bar{s} - c} v = \frac{1}{\bar{s} - c} (s - ce_i)v = 0,$$

contrary to our choice of  $v$ .

(ii) Now suppose  $s \in e_i A e_i$  satisfies  $sV = 0$ . Assume to the contrary that  $\bar{s} \notin \mathfrak{q}$ ; then  $\bar{s}^{-1} \in S_{\mathfrak{q}}$ . Whence

$$e_i V = \frac{s}{\bar{s}} e_i V = \frac{1}{\bar{s}} sV = 0,$$

contrary to our choice of vertex  $i$ . □

**Definition 6.4.** We say an element  $p \in e_j A_{\mathfrak{q}} e_i$  is *vertex invertible* if there is an element  $p^* \in e_i A_{\mathfrak{q}} e_j$  such that

$$p^* p = e_i \quad \text{and} \quad p p^* = e_j.$$

Denote by  $(e_j A_{\mathfrak{q}} e_i)^\circ$  the set of vertex invertible elements in  $e_j A_{\mathfrak{q}} e_i$ . For an arrow  $a \in Q_1^t$ , denote by  $\delta_a$  the unique arrow with  $h(\delta_a) = t(a)$ ; in particular,  $\delta_a \in Q_1^*$ .

**Lemma 6.5.** *A path  $p \in A$  is vertex invertible in  $A_{\mathfrak{q}}$  if and only if  $x_D \nmid \bar{p}$  and the leftmost arrow subpath of  $p$  is not an arrow  $\delta_a \in Q_1^*$  for which  $x_D \mid \bar{a}$ .*

*Proof.* (i) First suppose  $x_D \mid \bar{p}$ . Assume to the contrary that  $p$  has vertex inverse  $p^*$ ; then

$$(7) \quad p^* = \sum_{j=1}^m s_j^{-1} p_j$$

for some  $s_j \in S \setminus \mathfrak{q}$  and  $p_j \in e_{t(p)} A e_{h(p)}$ . In particular,

$$1 = \overline{p p^*} = \bar{p} \sum_j s_j^{-1} \bar{p}_j.$$

Whence

$$s_1 \cdots s_m = \bar{p} \sum_j (s_1 \cdots \hat{s}_j \cdots s_m) \bar{p}_j.$$

Thus  $x_D \mid s_1 \cdots s_m$  since  $x_D \mid \bar{p}$ . Therefore  $x_D \mid s_j$  for some  $j$ . But then  $s_j \in \mathfrak{q}$ , a contradiction to our choice of  $s_j$ .

(ii) Now suppose the leftmost arrow subpath of  $p$  is an arrow  $\delta_a \in Q_1^*$  for which  $x_D \mid \bar{a}$ . Again assume to the contrary that  $p$  has vertex inverse  $p^*$  given by (7). Since  $\deg^+ h(p) = 1$ , each nontrivial path  $q \in A$  with tail at  $h(p)$  satisfies  $x_D \mid \bar{q}$ . Thus  $x_D$  divides each  $\bar{p}_j$  since  $p_j \in e_{t(p)} A e_{h(p)}$  is a sum of nontrivial paths. But  $x_D$  does not divide any  $s_j$  since  $s_j \in S \setminus \mathfrak{q}$ . Therefore, since  $\bar{p} \in B$ ,

$$x_D \mid \overline{p^* p} = 1,$$

a contradiction.

(iii) Finally suppose  $x_D \nmid \bar{p}$  and the leftmost arrow subpath of  $\bar{p}$  is not an arrow  $\delta_a \in Q_1^*$  for which  $x_D \mid \bar{a}$ . Then there is a path  $q \in e_{t(p)}Ae_{h(p)}$  satisfying  $x_D \nmid \bar{q}$ , by Lemma 5.2. Whence  $pq$  is a cycle satisfying  $x_D \nmid \overline{pq}$ ; that is,  $\overline{pq} \in S \setminus \mathfrak{q}$ . Furthermore,  $q$  has a vertex subpath  $i$  for which  $e_iAe_i = Se_i$ , by Lemma 4.1.4. Thus

$$p^* := q(\overline{pq})^{-1}$$

is in  $A_{\mathfrak{q}}$ . But then

$$p^*p = \frac{q}{\overline{pq}}p = \frac{\overline{qp}}{\overline{pq}}e_{t(p)} = e_{t(p)} \quad \text{and} \quad pp^* = p\frac{q}{\overline{pq}} = e_{h(p)}\frac{\overline{pq}}{\overline{pq}} = e_{h(p)}.$$

Therefore  $p$  is vertex invertible in  $A_{\mathfrak{q}}$ . □

**Lemma 6.6.** *Let  $V$  be a simple  $A_{\mathfrak{q}}$ -module.*

- (1) *If  $a \in Q_1 \setminus Q_1^*$  satisfies  $x_D \mid \bar{a}$ , then  $aV = 0$ .*
- (2) *If  $\delta_a \in Q_1^*$  satisfies  $x_D \mid \bar{a}$ , then  $\delta_a V = 0$ .*

*Proof.* Let  $a \in Q_1$  be an arrow for which  $x_D \mid \bar{a}$ .

(i) First suppose  $a \in Q_1 \setminus (Q_1^* \cup Q_1^t)$ . We claim that  $aV = 0$ . Since  $a \in Q_1 \setminus (Q_1^* \cup Q_1^t)$ , there are paths

$$s \in e_{h(a)}Ae_{t(a)} \quad \text{and} \quad t \in e_{t(a)}Ae_{h(a)}$$

such that  $x_D \nmid \bar{s}$  and  $x_D \nmid \bar{t}$ , by Lemma 5.2. In particular,  $x_D \nmid \overline{st}$ . Whence

$$\overline{st} \in S \setminus \mathfrak{q}.$$

Thus

$$a = \frac{st}{\overline{st}}a = \frac{s}{\overline{st}}ta \in A_{\mathfrak{q}}\mathfrak{q}e_{t(a)}.$$

But  $ta \in \mathfrak{q}e_{t(a)} \cap e_{t(a)}Ae_{t(a)}$ . Therefore  $a$  annihilates  $V$  by Lemma 6.3.

(ii) Now suppose  $a \in Q_1^t$ . Set  $\delta := \delta_a \in Q_1^*$ .

(ii.a) We first claim that  $a\delta V = 0$ . By assumption (B),  $\deg^+ t(\delta) \geq 2$  and  $\deg^+ h(a) \geq 2$ . Thus there are paths

$$s \in e_{h(a)}Ae_{t(\delta)} \quad \text{and} \quad t \in e_{t(\delta)}Ae_{h(a)}$$

such that  $x_D \nmid \bar{s}$  and  $x_D \nmid \bar{t}$ , by Lemma 5.2. Whence

$$\overline{st} \in S \setminus \mathfrak{q}.$$

Thus

$$a\delta = \frac{st}{\overline{st}}a\delta = \frac{s}{\overline{st}}ta\delta \in A_{\mathfrak{q}}\mathfrak{q}e_{t(\delta)}.$$

Therefore  $a\delta$  annihilates  $V$  by Lemma 6.3.

(ii.b) We claim that  $aV = 0$ . If  $e_{t(a)}V = 0$ , then  $aV = 0$ , so suppose there is some nonzero  $v \in e_{t(a)}V$ . Assume to the contrary that  $av \neq 0$ . Since  $V$  is simple and  $\deg^+ t(a) = 1$ , there is some  $p \in A_{\mathfrak{q}}$  such that

$$w := \delta pav \in e_{t(a)}V$$



is nonzero. By Claim (2.i),  $aw = (a\delta)(pav) = 0$ . Thus  $A_{\mathfrak{q}}w$  is a submodule of  $e_{t(a)}V$ , since  $a$  is the only arrow with tail at  $t(a)$ , and  $\delta$  is not vertex invertible by Lemma 6.5. Therefore there is some  $c \in k^*$  such that  $cw = v$  since  $V$  is simple. But then

$$v = cw = c\delta p(av) = c\delta p(aw) = 0,$$

contrary to our choice of  $v$ .

(ii.c) Finally, we claim that  $\delta V = 0$ . Assume to the contrary that there is some  $v \in e_{t(\delta)}V$  such that  $\delta v \neq 0$ . By Claim (2.i),  $a\delta v = 0$ . But again  $a$  is the only arrow with tail at  $t(a)$ , and  $\delta$  is not vertex invertible by Lemma 6.5. Therefore  $V$  is not simple, a contradiction.  $\square$

For each  $\mathfrak{q}_D \in \text{Spec } S$  minimal over  $\mathfrak{m}_0$ , set

$$\epsilon_D := 1_A - \sum_{a \in Q_1^t : x_D | \bar{a}} e_{t(a)}.$$

**Theorem 6.7.** *Let  $\mathfrak{q} = \mathfrak{q}_D \in \text{Spec } S$  be minimal over  $\mathfrak{m}_0 \in \text{Max } R$ . Suppose there are  $n$  arrows  $a_1, \dots, a_n \in Q_1^t$  such that  $x_D \mid \bar{a}_\ell$ . Then there are precisely  $n + 1$  non-isomorphic simple  $A_{\mathfrak{q}}$ -modules:*

$$(8) \quad V_0 := A_{\mathfrak{q}}\epsilon_D / A_{\mathfrak{q}}\mathfrak{q}\epsilon_D \cong (S_{\mathfrak{q}}/\mathfrak{q})\epsilon_D,$$

and for each  $1 \leq \ell \leq n$ , a vertex simple

$$(9) \quad V_\ell := ke_{t(a_\ell)} \cong (R_{\mathfrak{m}_0}/\mathfrak{m}_0)e_{t(a_\ell)}.$$

*Proof.* Let  $V$  be a simple  $A_{\mathfrak{q}}$ -module. Let  $a \in Q_1^t$  be such that  $x_D \mid \bar{a}$ . Then either  $V$  is the vertex simple  $V = ke_{t(a)}$ , or  $e_{t(a)}$  annihilates  $V$ , by Lemma 6.6.

So suppose  $e_{t(a)}V = 0$  for each  $a \in Q_1^t$  satisfying  $x_D \mid \bar{a}$ . We claim that the sequence of left  $A_{\mathfrak{q}}$ -modules

$$0 \rightarrow A_{\mathfrak{q}}\mathfrak{q}\epsilon_D \rightarrow A_{\mathfrak{q}}\epsilon_D \xrightarrow{g} V \rightarrow 0$$

is exact.

We first claim that  $g$  is onto. Indeed, since  $V \neq 0$ , there is a vertex summand  $e_i$  of  $\epsilon_D$  for which  $e_iV \neq 0$ . Let  $e_j$  be an arbitrary vertex summand of  $\epsilon_D$ . Then there is a path  $p \in e_j A e_i$  satisfying  $x_D \nmid \bar{p}$ , by Lemma 5.2. Thus, since  $e_j$  is a summand of  $\epsilon_D$ ,  $p$  is vertex invertible by Lemma 6.5. Whence  $e_jV \neq 0$  since  $e_iV \neq 0$ . Therefore  $g$  is onto by Lemma 6.2.

We now claim that the kernel of  $g$  is  $A_{\mathfrak{q}}\mathfrak{q}\epsilon_D$ . Let  $b \in \epsilon_D A \epsilon_D$  be an arrow satisfying  $bV = 0$ . Then there is a path  $p \in e_{t(b)} A e_{h(b)}$  satisfying  $x_D \nmid \bar{p}$ , by Lemma 5.2. Thus, since  $e_{t(b)}$  and  $e_{h(b)}$  are vertex summands of  $\epsilon_D$ ,  $p$  is vertex invertible in  $A_{\mathfrak{q}}$ . Whence

$$b = (p^*p)b = p^*(pb) \in A_{\mathfrak{q}}\mathfrak{q}\epsilon_D.$$

Thus the  $A_{\mathfrak{q}}\epsilon_D$ -annihilator of  $V$  is  $A_{\mathfrak{q}}\mathfrak{q}\epsilon_D$ , by Lemma 6.2.

Therefore  $V = V_0$ . The simple modules  $V_0, \dots, V_n$  exhaust the possible simple  $A_{\mathfrak{q}}$ -modules, again by Lemma 6.2.  $\square$

If  $p \in A_{\mathbf{q}}$  is a concatenation of paths and vertex inverses of paths in  $A$ , then we call  $p$  a *path*.

**Lemma 6.8.** *Suppose  $i \in Q_0$  satisfies  $e_i \epsilon_D \neq 0$ . Then for each  $j \in Q_0$ , the corner rings  $e_j A_{\mathbf{q}} e_i$  and  $e_i A_{\mathbf{q}} e_j$  are cyclic free  $S_{\mathbf{q}}$ -modules. Consequently,  $A_{\mathbf{q}} e_i$  and  $e_i A_{\mathbf{q}}$  are free  $S_{\mathbf{q}}$ -modules.*

*Proof.* Suppose  $e_i$  is a vertex summand of  $\epsilon_D$ . Then either  $e_i A e_i = S e_i$ , or  $i = t(a)$  for some  $a \in Q_1^t$  with  $x_D \nmid \bar{a}$ , by Lemma 4.1.4. In the latter case,  $a$  is vertex invertible by Lemma 6.5. Thus in either case we have

$$e_i A_{\mathbf{q}} e_i = S_{\mathbf{q}} e_i.$$

Therefore  $A_{\mathbf{q}} e_i$  and  $e_i A_{\mathbf{q}}$  are  $S_{\mathbf{q}}$ -modules.

(i) We claim that for each  $j \in Q_0$ ,  $e_j A_{\mathbf{q}} e_i$  is generated as an  $S_{\mathbf{q}}$ -module by a single path; a similar argument holds for  $e_i A_{\mathbf{q}} e_j$ .

(i.a) First suppose  $j$  is not the tail of an arrow  $a \in Q_1^t$  for which  $x_D \mid \bar{a}$ . Since  $D \in \mathcal{S}'$  is a simple matching of  $Q'$ , there is path  $s$  from  $i$  to  $j$  for which  $x_D \nmid \bar{s}$  (that is,  $\psi(s)$  is supported on  $Q' \setminus D$ ). Thus  $s$  has a vertex inverse  $s^* \in e_i A_{\mathbf{q}} e_j$ , by Lemma 6.5.

Let  $t \in e_j A_{\mathbf{q}} e_i$  be arbitrary. Then  $s^* t$  is in  $e_i A_{\mathbf{q}} e_i = S_{\mathbf{q}} e_i$ . Whence

$$t = s s^* t \in s S_{\mathbf{q}}.$$

Therefore  $e_j A_{\mathbf{q}} e_i = s S_{\mathbf{q}}$ .

(i.b) Now suppose  $j$  is the tail of an arrow  $a \in Q_1^t$  for which  $x_D \mid \bar{a}$ ; in particular,  $j \neq i$ . Since  $D \in \mathcal{S}'$  is a simple matching of  $Q'$ , there is path  $s$  from  $i$  to  $t(\delta_a)$  for which  $x_D \nmid \bar{s}$ . Thus  $s$  has a vertex inverse  $s^* \in e_i A_{\mathbf{q}} e_{t(\delta_a)}$ , again by Lemma 6.5.

Let  $t \in e_j A_{\mathbf{q}} e_i$  be arbitrary. Since  $j \neq i$  and  $\deg^+ j = 1$ , there is some  $r \in e_{t(\delta_a)} A_{\mathbf{q}} e_i$  satisfying  $t = \delta_a r$ . Whence

$$t = \delta_a r = \delta_a s s^* r \in \delta_a s S_{\mathbf{q}}.$$

Therefore  $e_j A_{\mathbf{q}} e_i = \delta_a s S_{\mathbf{q}}$ .

(ii) Finally, we claim that  $e_j A_{\mathbf{q}} e_i$  is a free  $S_{\mathbf{q}}$ -module; a similar argument holds for  $e_i A_{\mathbf{q}} e_j$ . By Claim (i), there is a path  $s$  such that

$$e_j A_{\mathbf{q}} e_i = s S_{\mathbf{q}}.$$

Furthermore, the  $S_{\mathbf{q}}$ -module homomorphism

$$S_{\mathbf{q}} \rightarrow s S_{\mathbf{q}}, \quad t \mapsto st,$$

is an isomorphism since  $S_{\mathbf{q}}$  and  $\bar{s}$  belong to the domain  $\text{Frac } B$ , and  $\bar{\tau}_{\psi}$  is injective.  $\square$

**Lemma 6.9.** *The  $A_{\mathbf{q}}$ -module  $V_0$  satisfies*

$$\text{pd}_{A_{\mathbf{q}}}(V_0) \leq \text{pd}_{S_{\mathbf{q}}}(S_{\mathbf{q}}/\mathbf{q}).$$

*Proof.* Consider a minimal free resolution of  $S_{\mathfrak{q}}/\mathfrak{q}$  over  $S_{\mathfrak{q}}$ ,

$$\cdots \rightarrow S_{\mathfrak{q}}^{\oplus n_1} \rightarrow S_{\mathfrak{q}} \rightarrow S_{\mathfrak{q}}/\mathfrak{q} \rightarrow 0.$$

Set  $\epsilon := \epsilon_D$ . By Lemma 6.8,  $A_{\mathfrak{q}}\epsilon$  is a free  $S_{\mathfrak{q}}$ -module. Thus  $A_{\mathfrak{q}}\epsilon$  is a flat  $S_{\mathfrak{q}}$ -module, that is, the functor  $A_{\mathfrak{q}}\epsilon \otimes_{S_{\mathfrak{q}}} -$  is exact. Therefore the sequence of left  $A_{\mathfrak{q}}$ -modules

$$(10) \quad \cdots \rightarrow A_{\mathfrak{q}}\epsilon \otimes S_{\mathfrak{q}}^{\oplus n_1} \rightarrow A_{\mathfrak{q}}\epsilon \otimes S_{\mathfrak{q}} \rightarrow A_{\mathfrak{q}}\epsilon \otimes S_{\mathfrak{q}}/\mathfrak{q} \rightarrow 0$$

is exact. Each term is a projective  $A_{\mathfrak{q}}$ -module since

$$A_{\mathfrak{q}}\epsilon \otimes_{S_{\mathfrak{q}}} (S_{\mathfrak{q}}^{\oplus n_i}) \cong (A_{\mathfrak{q}}\epsilon)^{\oplus n_i}.$$

Furthermore, there is a left  $A_{\mathfrak{q}}$ -module isomorphism

$$V_0 = A_{\mathfrak{q}}\epsilon/A_{\mathfrak{q}}\mathfrak{q}\epsilon \cong A_{\mathfrak{q}}\epsilon \otimes_{S_{\mathfrak{q}}} S_{\mathfrak{q}}/\mathfrak{q}.$$

Therefore (10) is a projective resolution of  $V_0$  over  $A_{\mathfrak{q}}$  of length at most  $\text{pd}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q})$ .  $\square$

**Lemma 6.10.** *The local ring  $S_{\mathfrak{q}}$  is regular.*

*Proof.*  $S$  is normal since  $S$  is isomorphic to the center of the (noetherian) NCCR  $A'$ . In particular, the singular locus of  $\text{Max } S$  has codimension at least 2. Furthermore, the zero locus  $\mathcal{Z}(\mathfrak{q})$  in  $\text{Max } S$  has codimension 1, by Theorem 5.7.1. Therefore  $\mathcal{Z}(\mathfrak{q})$  contains a smooth point of  $\text{Max } S$ .  $\square$

**Proposition 6.11.** *Let  $\mathfrak{q} \in \text{Spec } S$  be minimal over  $\mathfrak{m}_0$ . Then each simple  $A_{\mathfrak{q}}$ -module has projective dimension 1. Consequently, for each simple  $A_{\mathfrak{q}}$ -module  $V$ ,*

$$\text{pd}_{A_{\mathfrak{q}}}(V) = \text{ht}_S(\mathfrak{q}).$$

*Proof.* Recall the classification of simple  $A_{\mathfrak{q}}$ -modules given in Theorem 6.7.

(i) Let  $V_0$  be the simple  $A_{\mathfrak{q}}$ -module defined in (8). Then

$$1 \stackrel{(i)}{\leq} \text{pd}_{A_{\mathfrak{q}}}(V_0) \stackrel{(ii)}{\leq} \text{pd}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q}) \stackrel{(iii)}{=} \text{ht}_S(\mathfrak{q}) \stackrel{(iv)}{=} 1.$$

Indeed, (i) holds since  $V_0$  is clearly not a direct summand of a free  $A_{\mathfrak{q}}$ -module; (ii) holds by Lemma 6.9; (iii) holds by Lemma 6.10; and (iv) holds by Theorem 5.7.1.

(ii) Fix  $1 \leq \ell \leq n$ , and let  $V_{\ell}$  be the vertex simple  $A_{\mathfrak{q}}$ -module defined in (9). Set  $a := a_{\ell}$ . We claim that  $V_{\ell}$  has minimal projective resolution

$$(11) \quad 0 \rightarrow A_{\mathfrak{q}}e_{h(a)} \xrightarrow{\cdot a} A_{\mathfrak{q}}e_{t(a)} \xrightarrow{\cdot 1} ke_{t(a)} = V_{\ell} \rightarrow 0.$$

(ii.a) We first claim that  $\cdot a$  is injective. Suppose  $b \in A_{\mathfrak{q}}e_{h(a)}$  is nonzero. Then  $\bar{\tau}_{\psi}(ba) = \bar{b} \cdot \bar{a} \neq 0$  since  $B$  is an integral domain. Whence  $ba \neq 0$  since  $\bar{\tau}_{\psi}$  is injective. Therefore  $\cdot a$  is injective.

(ii.b) We now claim that  $\text{im}(\cdot a) = \ker(\cdot 1)$ . Since  $aV = 0$ , we have  $\text{im}(\cdot a) \subseteq \ker(\cdot 1)$ . To show the reverse inclusion, suppose  $g \in \ker(\cdot 1)$ ; then  $gV = 0$ . We may write

$$g = \sum_j s_j^{-1} p_j,$$

where each  $p_j \in Ae_{t(a)}$  is a path and  $s_j \in S \setminus \mathfrak{q}$ . If  $p_j$  is nontrivial, then  $p_j = p'_j a$  for some path  $p'_j$  since  $\deg^+ t(a) = 1$ . Whence

$$p_j V_\ell = p'_j a V_\ell = 0.$$

It thus suffices to suppose that each  $p_j$  is trivial,  $p_j = e_{t(a)}$ . But then  $g = s^{-1}e_{t(a)}$  for some  $s \in S \setminus \mathfrak{q}$ . Therefore

$$e_{t(a)} V_\ell = sg V_\ell = 0,$$

a contradiction.

(ii.c) Finally, (11) is minimal since  $V_\ell$  is clearly not a direct summand of a free  $A_{\mathfrak{q}}$ -module.  $\square$

Lemmas 6.12, 6.14, and Proposition 6.13 are not specific to homotopy algebras.

**Lemma 6.12.** *Suppose  $S$  is a depiction of  $R$ . Let  $\mathfrak{p} \in \text{Spec } R$  and  $\mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p})$ . If  $\text{ht}_S(\mathfrak{q}) = 1$ , then  $\text{ght}_R(\mathfrak{p}) = 1$ .*

*Proof.* Assume to the contrary that  $\text{ght}_R(\mathfrak{p}) = 0$ . Then there is a depiction  $S'$  of  $R$  and a prime ideal  $\mathfrak{q}' \in \iota_{S'/R}(\mathfrak{p})$  such that  $\text{ht}_{S'}(\mathfrak{q}') = 0$ . Whence  $\mathfrak{q}' = 0$  since  $S'$  is an integral domain. But then  $\mathfrak{q}' \cap R = 0 \neq \mathfrak{q} \cap R = \mathfrak{p}$ , a contradiction. Therefore

$$\text{ht}_S(\mathfrak{q}) = 1 \leq \text{ght}_R(\mathfrak{p}) \leq \text{ht}_S(\mathfrak{q}).$$

$\square$

Recall that an ideal  $I$  of an integral domain  $S$  is a projective  $S$ -module if and only if  $I$  is invertible, i.e., there is a fractional ideal  $J$  such that  $IJ = S$ . In this case,  $I$  is a finitely generated rank one  $S$ -module (e.g., [C, Theorem 19.10]).

**Proposition 6.13.** *Let  $B$  be an integral domain, and let  $A = [A^{ij}] \subset M_d(B)$  be a tiled matrix ring with cycle algebra  $S$ . For each  $j \in Q_0 := \{1, \dots, d\}$ , set  $e_j := E_{jj}$ . Suppose that*

- (1)  $S$  is a regular local ring.
- (2) *There is some  $i \in Q_0$  such that*
  - (a)  $A^i = S$ ;
  - (b) *for each  $j \in Q_0$ ,  $A^{ij}$  is an invertible ideal of  $S$ ; and*
  - (c) *for each  $j \in Q_0$ , either  $(e_i A e_j)^\circ \neq \emptyset$ , or there is some  $\ell \in Q_0$  and  $b \in e_j A e_\ell$  satisfying*

$$e_j A = bA \oplus k e_j \quad \text{and} \quad (e_i A e_\ell)^\circ \neq \emptyset.$$

*Then*

$$\text{gldim } A \leq \dim S.$$

*Proof.* Denote by  $e_i$  the diagonal matrix  $E_{ii}$ . Suppose the hypotheses hold, and set  $n := \dim S$ . Let  $V$  be a left  $A$ -module. We claim that

$$\text{pd}_A(V) \leq n.$$

It suffices to show that there is a projective resolution  $P_\bullet$  of  $V$ ,

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} V \rightarrow 0,$$

for which  $\ker \delta_{n-1}$  is a projective  $A$ -module [R, Proposition 8.6.iv].

(i) We first claim that there is a projective resolution  $P_\bullet$  of  $V$  so that for each  $\alpha \geq 1$ ,

$$(12) \quad \ker \delta_\alpha = Ae_i \ker \delta_\alpha.$$

Indeed, fix  $j \in Q_0$ , and recall assumption (2.c). If  $p \in (e_i Ae_j)^\circ$ , then

$$e_j \ker \delta_\alpha = p^* p \ker \delta_\alpha = p^* e_i p \ker \delta_\alpha \subseteq Ae_i \ker \delta_\alpha.$$

Otherwise there is some  $\ell \in Q_0$  and  $b \in e_j Ae_\ell$  such that  $e_j A = bA \oplus ke_j$  and  $(e_i Ae_\ell)^\circ \neq \emptyset$ . Let  $p \in (e_i Ae_\ell)^\circ$ . Since the sum  $e_j A = bA \oplus ke_j$  is direct, we may assume that for each  $\alpha \geq 1$ ,

$$\delta_\alpha|_{e_j P_\alpha} = b \cdot \delta_\alpha|_{e_\ell P_\alpha}.$$

Furthermore, for nonzero  $q \in e_\ell A$ ,  $bq \neq 0$  since  $B$  is an integral domain. Thus

$$e_j \ker \delta_\alpha = b \ker \delta_\alpha.$$

Whence

$$e_j \ker \delta_\alpha = b \ker \delta_\alpha = bp^* e_i p \ker \delta_\alpha \subseteq Ae_i \ker \delta_\alpha.$$

Therefore in either case,

$$e_j \ker \delta_\alpha \subseteq Ae_i \ker \delta_\alpha.$$

(ii) Fix a projective resolution  $P_\bullet$  of  $V$  satisfying (12). We claim that the left  $A$ -module  $Ae_i \ker \delta_{n-1}$  is projective.

The right  $A$ -module  $e_i A$  is projective, hence flat. Thus, setting  $\otimes := \otimes_A$ , the complex of  $S$ -modules

$$(13) \quad \cdots \longrightarrow e_i A \otimes P_2 \xrightarrow{1 \otimes \delta_2} e_i A \otimes P_1 \xrightarrow{1 \otimes \delta_1} e_i A \otimes P_0 \xrightarrow{1 \otimes \delta_0} e_i A \otimes V \rightarrow 0$$

is exact. Each term  $e_i A \otimes P_\ell$  is a free  $S$ -module since

$$\begin{aligned} e_i A \otimes P_\ell &\cong e_i A \otimes \bigoplus_j (Ae_j)^{\oplus n_j} \cong \bigoplus_j (e_i A \otimes Ae_j)^{\oplus n_j} \\ &\cong \bigoplus_j (e_i Ae_j)^{\oplus n_j} \cong \bigoplus_j (A^{ij})^{\oplus n_j} \stackrel{(I)}{\cong} \bigoplus_j S^{\oplus n_j}, \end{aligned}$$

where (I) holds by assumption (2.b). Furthermore,  $e_i A \otimes V$  is an  $S$ -module since  $e_i Ae_i \cong S$  by assumption (2.a). Therefore (13) is a free resolution of an  $S$ -module. But  $\text{gldim } S = \dim S = n$  by assumption (1). Therefore the  $n$ th syzygy module of (13) is a free  $S$ -module,

$$\ker(1 \otimes \delta_{n-1}) \cong S^{\oplus m}.$$

Since  $e_i A$  is a flat right  $A$ -module, the sequence

$$0 \rightarrow e_i A \otimes \ker \delta_{n-1} \longrightarrow e_i A \otimes P_{n-1} \xrightarrow{1 \otimes \delta_{n-1}} e_i A \otimes P_{n-2}$$

is exact. Whence

$$e_i A \otimes \ker \delta_{n-1} \cong \ker(1 \otimes \delta_{n-1}) \cong S^{\oplus m}.$$

Therefore

$$Ae_i \ker \delta_{n-1} \cong Ae_i A \otimes \ker \delta_{n-1} \cong Ae_i S^{\oplus m} \stackrel{(I)}{\cong} A(e_i Ae_i)^{\oplus m} \cong (Ae_i)^{\oplus m},$$

where (I) holds by assumption (2.a), proving our claim.

(iii) Finally,  $\ker \delta_{n-1}$  is a projective left  $A$ -module by Claims (i) and (ii). Therefore  ${}_A V$  has projective dimension at most  $n$ .  $\square$

**Lemma 6.14.** *Suppose  $S$  is an integral domain and a  $k$ -algebra, and  $R$  is a subalgebra of  $S$ . Let  $\mathfrak{p} \in \text{Spec } R$ . If  $\mathfrak{t} \in \text{Spec}(SR_{\mathfrak{p}})$  is a minimal prime over  $\mathfrak{p}R_{\mathfrak{p}}$ , then the ideal  $\mathfrak{t} \cap S \in \text{Spec } S$  is a minimal prime over  $\mathfrak{p}$ .*

*Proof.* Suppose that  $\mathfrak{t} \cap S$  is not a minimal prime over  $\mathfrak{p}$ . We claim that  $\mathfrak{t}$  is not a minimal prime over  $\mathfrak{p}R_{\mathfrak{p}}$ . Indeed, since  $\mathfrak{t} \cap S$  is not minimal, there is some  $\mathfrak{q} \in \text{Spec } S$ , minimal over  $\mathfrak{p}$ , such that

$$\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{t} \cap S.$$

Fix  $a \in \mathfrak{t} \cap S \setminus \mathfrak{q}$ , and assume to the contrary that  $a \in \mathfrak{q}R_{\mathfrak{p}}$ . Then there is some  $b \in \mathfrak{q}$  and  $c \in R \setminus \mathfrak{p}$  such that  $a = bc^{-1}$ . In particular,  $ac = b \in \mathfrak{q}$ . Whence  $c \in \mathfrak{q}$  since  $c \in R \subseteq S$  and  $\mathfrak{q}$  is prime. Thus

$$c \in \mathfrak{q} \cap R \stackrel{(I)}{=} \mathfrak{p},$$

where (I) holds since  $\mathfrak{q}$  is a minimal prime over  $\mathfrak{p}$ . But  $c \notin \mathfrak{p}$ , a contradiction. Whence  $a \in \mathfrak{t} \setminus \mathfrak{q}R_{\mathfrak{p}}$ . Thus

$$\mathfrak{p}R_{\mathfrak{p}} \subseteq \mathfrak{q}R_{\mathfrak{p}} \subset \mathfrak{t}.$$

Furthermore,  $\mathfrak{q}R_{\mathfrak{p}}$  is a prime ideal of  $SR_{\mathfrak{p}}$ . Therefore  $\mathfrak{t}$  is not a minimal prime over  $\mathfrak{p}$ .  $\square$

Again let  $A$  be a nonnoetherian homotopy algebra satisfying assumptions (A) and (B). Recall that the center and cycle algebra of  $A_{\mathfrak{m}_0} := A \otimes_R R_{\mathfrak{m}_0}$  are isomorphic to  $R_{\mathfrak{m}_0}$  and  $SR_{\mathfrak{m}_0}$  respectively.

**Theorem 6.15.**  *$A_{\mathfrak{m}_0}$  is a noncommutative desingularization of its center. Furthermore, for each  $\mathfrak{t} \in \text{Spec}(SR_{\mathfrak{m}_0})$  minimal over  $\mathfrak{t} \cap R_{\mathfrak{m}_0}$ ,*

$$\text{gldim } A_{\mathfrak{t}} = \dim(SR_{\mathfrak{m}_0})_{\mathfrak{t}} = \dim S_{\mathfrak{t} \cap S}.$$

*Proof.* By Lemma 6.14 (with  $\mathfrak{p} = \mathfrak{m}_0$ ), it suffices to consider prime ideals  $\mathfrak{q} \in \text{Spec } S$  which are minimal over  $\mathfrak{m}_0$ .

(i)  $A_{\mathfrak{m}_0}$  is cycle regular. Let  $\mathfrak{q} \in \text{Spec } S$  be minimal over  $\mathfrak{m}_0$ , and let  $V$  be a simple  $A_{\mathfrak{q}}$ -module. The hypotheses of Proposition 6.13 hold: condition (1) holds by Lemma

6.10; (2.a) holds by Lemma 4.1.4; (2.b) holds by Lemma 6.8; and (2.c) holds by Lemma 6.5. Thus

$$1 \stackrel{(i)}{\leq} \text{gldim } A_{\mathfrak{q}} \stackrel{(ii)}{\leq} \dim S_{\mathfrak{q}} = \text{ht}_S(\mathfrak{q}) \stackrel{(iii)}{=} 1 \stackrel{(iv)}{=} \text{ght}_R(\mathfrak{m}_0) \stackrel{(v)}{=} \text{pd}_{A_{\mathfrak{q}}}(V).$$

Indeed, (i) and (v) hold by Proposition 6.11; (ii) holds by Proposition 6.13; (iii) holds by Theorem 5.7.3; and (iv) holds by Lemma 6.12. Therefore  $A_{\mathfrak{m}_0}$  is cycle regular.

(ii)  $A_{\mathfrak{m}_0}$  is a noncommutative desingularization. By [B4, Corollary 2.14.1] the (noncommutative) function fields of  $A$  and  $R$ , and hence  $A_{\mathfrak{m}_0}$  and  $R_{\mathfrak{m}_0}$ , are Morita equivalent,

$$A \otimes_R \text{Frac } R \sim \text{Frac } R.$$

(iii) Finally, suppose  $\mathfrak{q} \in \text{Spec } S$  is minimal over  $\mathfrak{q} \cap R$ . We claim that  $\text{gldim } A_{\mathfrak{q}} = \dim S_{\mathfrak{q}}$ . By Theorem 5.7.2, either  $\mathfrak{q} = \mathfrak{q}_D$  for some  $D \in S'$ , or  $\mathfrak{q} = 0$ . The case  $\mathfrak{q} = \mathfrak{q}_D$  was shown in Claim (i), so suppose  $\mathfrak{q} = 0$ .

We first claim that for each  $i \in Q_0$ ,

$$(14) \quad e_i A_{\mathfrak{q}} e_i = (\text{Frac } S) e_i.$$

Indeed, let  $g \in \text{Frac } S$  be arbitrary. Fix  $j \in Q_0$  for which  $e_j A e_j = S e_j$ . Since  $S$  is a domain,

$$(15) \quad e_j A_{\mathfrak{q}} e_j = S_{\mathfrak{q}} e_j = (\text{Frac } S) e_j.$$

Thus there is an element  $s \in e_j A_{\mathfrak{q}} e_j$  satisfying  $\bar{s} = g$ .

Now fix a cycle  $t_2 e_j t_1 \in e_i A_{\mathfrak{q}} e_i$  that passes through  $j$ . Then  $t_1 t_2 \in e_j A_{\mathfrak{q}} e_j$  has a vertex inverse  $(t_1 t_2)^*$  by (15). Thus the element

$$s' := t_2 (t_1 t_2)^* s t_1 \in e_i A_{\mathfrak{q}} e_i$$

satisfies  $\bar{s}' = \bar{s} = g$ . Therefore (14) holds.

We now claim that for each  $i, j \in Q_0$ , there is a  $(\text{Frac } S)$ -module isomorphism<sup>8</sup>

$$(16) \quad e_j A_{\mathfrak{q}} e_i \cong \text{Frac } S.$$

Let  $s \in e_j A_{\mathfrak{q}} e_i$  be arbitrary, and fix a cycle  $t_2 e_j t_1 \in e_i A_{\mathfrak{q}} e_i$  that passes through  $j$ . Then  $t_1 t_2$  has a vertex inverse  $(t_1 t_2)^*$  by (14). Thus

$$s = (t_1 t_2)^* s (t_2 t_1) \in (\text{Frac } S) t_1.$$

Whence  $e_j A_{\mathfrak{q}} e_i \subseteq (\text{Frac } S) t_1$ . Conversely, (14) implies  $e_j A_{\mathfrak{q}} e_i \supseteq (\text{Frac } S) t_1$ . Thus

$$e_j A_{\mathfrak{q}} e_i = (\text{Frac } S) t_1.$$

Furthermore, the  $(\text{Frac } S)$ -module homomorphism

$$\text{Frac } S \rightarrow (\text{Frac } S) t_1, \quad s \mapsto s t_1,$$

is an isomorphism since  $\bar{t}_1$  and  $\text{Frac } S$  are in the domain  $\text{Frac } B$ , and  $\bar{\tau}_\psi$  is injective. Therefore (16) holds.

<sup>8</sup>In general,  $\bar{\tau}_\psi(e_j A e_i)$  is not contained in  $\text{Frac } S$ ; otherwise (16) would trivially hold.

It follows from (14) and (16) that

$$A_{\mathfrak{q}} \cong M_d(\text{Frac } S).$$

Thus  $A_{\mathfrak{q}}$  is a semisimple algebra. Therefore

$$\text{gldim } A_{\mathfrak{q}} = 0 = \dim(\text{Frac } S) = \dim S_{\mathfrak{q}}.$$

□

## 7. LOCAL ENDOMORPHISM RINGS

Fix a simple matching  $D \in \mathcal{S}'$  for which  $\mathfrak{q} := \mathfrak{q}_D$  is a minimal prime over  $\mathfrak{m}_0$ . For  $a \in Q_1$ , recall the ideal

$$\mathfrak{m}_a := \bar{\tau}_{\psi}(e_{t(a)}Aa) \subset S$$

from Proposition 5.5. Set

$$\mathfrak{m}_D := \bigcap_{a \in Q_1^t : x_D | \bar{a}} \mathfrak{m}_a \quad \text{and} \quad \tilde{R} := (k + \mathfrak{m}_D)_{\mathfrak{m}_D} + \mathfrak{q}S_{\mathfrak{q}}.$$

**Lemma 7.1.** *Let  $a \in Q_1$ . If  $x_D \mid \bar{a}$ , then*

$$\mathfrak{m}_a S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}} = \sigma S_{\mathfrak{q}}.$$

We note that the relation  $\mathfrak{m}_a S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$  is nontrivial since if  $\bar{a} \neq x_D$ , then  $\mathfrak{q} \not\subseteq \mathfrak{m}_a$  in general; that is, there may be a cycle  $s$  for which  $x_D \mid \bar{s}$  but  $\bar{a} \nmid \bar{s}$ .

*Proof.* Suppose  $x_D \mid \bar{a}$ . Then

$$\sigma S_{\mathfrak{q}} \subseteq \bar{\tau}_{\psi}(e_{t(a)}Aa)S_{\mathfrak{q}} = \mathfrak{m}_a S_{\mathfrak{q}} \subseteq \mathfrak{q}S_{\mathfrak{q}} \stackrel{(1)}{=} \sigma S_{\mathfrak{q}},$$

where (1) holds by Proposition 5.6. □

**Proposition 7.2.** *The center  $Z(A_{\mathfrak{q}})$  of  $A_{\mathfrak{q}}$  is isomorphic to the subalgebra*

$$\tilde{R} := (k + \mathfrak{m}_D)_{\mathfrak{m}_D} + \mathfrak{q}S_{\mathfrak{q}} = \bigcap_{a \in Q_1^t} \bar{\tau}_{\psi}(e_{t(a)}A_{\mathfrak{q}}e_{t(a)}) \subset S_{\mathfrak{q}} \cong Z(A'_{\mathfrak{q}}).$$

*Proof.* Set

$$Q_1^t \cap D := Q_1^t \cap \psi^{-1}(D) = \{a \in Q_1^t : x_D \mid \bar{a}\}.$$



We claim that

$$\begin{aligned}
Z(A_{\mathfrak{q}}) &\stackrel{(I)}{\cong} \bigcap_{i \in Q_0} \bar{\tau}_{\psi}(e_i A_{\mathfrak{q}} e_i) \\
&\stackrel{(II)}{=} \bigcap_{a \in Q_1^t} \bar{\tau}_{\psi}(e_{t(a)} A_{\mathfrak{q}} e_{t(a)}) \\
&\stackrel{(III)}{=} \bigcap_{a \in Q_1^t} ((k + \mathfrak{m}_a)_{\mathfrak{q} \cap (k + \mathfrak{m}_a)} + \mathfrak{m}_a S_{\mathfrak{q}}) \\
&\stackrel{(IV)}{=} \bigcap_{a \in Q_1^t \cap D} ((k + \mathfrak{m}_a)_{\mathfrak{m}_a} + \mathfrak{q} S_{\mathfrak{q}}) \\
&\stackrel{(V)}{=} \bigcap_{a \in Q_1^t \cap D} (k + \mathfrak{m}_a)_{\mathfrak{m}_a} + \mathfrak{q} S_{\mathfrak{q}} \\
&= (k + \bigcap_{a \in Q_1^t \cap D} \mathfrak{m}_a) ((k + \bigcap_{a \in Q_1^t \cap D} \mathfrak{m}_a) \setminus \bigcup_{a \in Q_1^t \cap D} \mathfrak{m}_a)^{-1} + \mathfrak{q} S_{\mathfrak{q}} \\
&= (k + \mathfrak{m}_D)_{\mathfrak{m}_D} + \mathfrak{q} S_{\mathfrak{q}} \\
&= \tilde{R}.
\end{aligned}$$

Indeed, (I) holds by Lemma 4.1.2 and (II) holds by Lemma 4.1.4.

To show (III), suppose  $a \in Q_1^t$ . Recall the notation  $A^i := \bar{\tau}_{\psi}(e_i A e_i)$ . Then

$$A^{t(a)} = k + \mathfrak{m}_a \quad \text{and} \quad A^{h(a)} = S.$$

Thus by the definition of cyclic localization,

$$\begin{aligned}
\bar{\tau}_{\psi}(e_{t(a)} A_{\mathfrak{q}} e_{t(a)}) &= A_{\mathfrak{q} \cap A^{t(a)}}^{t(a)} + \sum_{\substack{qp \in e_{t(a)} A e_{t(a)} \\ \text{a nontrivial cycle}}} \bar{q} A_{\mathfrak{q} \cap A^{h(p)}}^{h(p)} \bar{p} \\
&= (k + \mathfrak{m}_a)_{\mathfrak{q} \cap (k + \mathfrak{m}_a)} + \sum_{\substack{q \in e_{t(a)} A e_{h(a)} \\ \text{a path}}} \bar{q} S_{\mathfrak{q}} \bar{a} \\
&= (k + \mathfrak{m}_a)_{\mathfrak{q} \cap (k + \mathfrak{m}_a)} + \mathfrak{m}_a S_{\mathfrak{q}}.
\end{aligned}$$

To show (IV), note that for  $a \in Q_1^t$ ,

$$\mathfrak{m}_a \subseteq \mathfrak{q} \quad \text{if and only if} \quad a \in \psi^{-1}(D).$$

Furthermore, if  $\mathfrak{m}_a \subseteq \mathfrak{q}$ , then  $\mathfrak{m}_a S_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}}$  by Lemma 7.1. Otherwise if  $\mathfrak{m}_a \not\subseteq \mathfrak{q}$ , then  $\mathfrak{m}_a S_{\mathfrak{q}} = S_{\mathfrak{q}}$ .

Finally, (V) holds since for  $a \in Q_1^t \cap \psi^{-1}(D)$ ,

$$\mathfrak{m}_a(k + \mathfrak{m}_a)_{\mathfrak{m}_a} \subseteq \mathfrak{q} S_{\mathfrak{q}}.$$

□

**Definition 7.3.** We say two arrows  $a, b \in Q_1$  are *coprime* if  $\bar{a}$  and  $\bar{b}$  are coprime in  $B$ ; that is, the only common factors of  $\bar{a}$  and  $\bar{b}$  in  $B$  are the units.

**Lemma 7.4.** Suppose the arrows in  $Q_1^t$  are pair-wise coprime, and let  $a \in Q_1^t$ . Consider a simple matching  $D \in \mathcal{S}'$  for which  $x_D \mid \bar{a}$ . Set  $\mathfrak{q} := \mathfrak{q}_D$  and  $i := t(a)$ . Then

$$Z(A_{\mathfrak{q}}) = \tilde{R} \mathbf{1} = A_{\mathfrak{q}}^i \mathbf{1} \cong e_i A_{\mathfrak{q}} e_i.$$

*Proof.* Suppose the arrows in  $Q_1^t$  are pair-wise coprime. Then each arrow in  $Q_1^t \setminus \{a\}$  is vertex invertible in  $A_q$  by Lemma 6.5. Thus for each  $j \in Q_0 \setminus \{i\}$ ,

$$e_j A_q e_j = S_q e_j,$$

by Lemma 4.1.4. The lemma then follows by Proposition 7.2.  $\square$

**Lemma 7.5.** *Let  $B$  be an integral domain, and let  $A = [A^{ij}] \subset M_d(B)$  be a tiled matrix ring. For each  $j \in Q_0 := \{1, \dots, d\}$ , set  $e_j := E_{jj}$ . Suppose  $i, j, k \in Q_0$  satisfy*

- (1)  $A^{ij} \neq 0$  and  $A^{ji} \neq 0$ .
- (2)  $A^i \mathbf{1}_d = Z(A)$ .
- (3) *There is some  $p \in e_j A e_i$  (resp.  $p \in e_i A e_j$ ) such that for each  $q \in e_k A e_i$  (resp.  $q \in e_i A e_k$ ), there is some  $r \in e_k A e_j$  satisfying  $rp = q$  (resp.  $pr = q$ ).*

Then

$$\mathrm{Hom}_{Z(A)}(e_j A e_i, e_k A e_i) \cong e_k A e_j \quad \text{and} \quad \mathrm{Hom}_{Z(A)}(e_i A e_j, e_i A e_k) \cong e_j A e_k.$$

*Proof.* Let  $f \in \mathrm{Hom}_{Z(A)}(e_j A e_i, e_k A e_i)$ . By assumption (1), there is some nonzero  $q \in e_i A e_j$ . By assumption (2), for  $p_1, p_2 \in e_j A e_i$ ,

$$\bar{q} \bar{p}_1 f(p_2) = \bar{p}_1 \bar{q} f(p_2) = f((p_1 q) p_2) = f(p_1 (q p_2)) = f((p_2 q) p_1) = \bar{p}_2 \bar{q} f(p_1) = \bar{q} \bar{p}_2 f(p_1).$$

Thus, since  $B$  is an integral domain,

$$\bar{p}_1 f(p_2) = \bar{p}_2 f(p_1).$$

In particular, if  $p_1$  and  $p_2$  are nonzero, then

$$\frac{\overline{f(p_1)}}{\bar{p}_1} = \frac{\overline{f(p_2)}}{\bar{p}_2} =: h \in \mathrm{Frac} B.$$

Therefore for each  $p \in e_j A e_i$ ,

$$(17) \quad \overline{f(p)} = h \bar{p}.$$

Let  $p_1 \in e_j A e_i$  be as in assumption (3). Then there is some  $r \in e_k A e_j$  such that

$$f(p_1) = r p_1.$$

Whence  $\bar{r} = h$  by (17), since  $B$  is an integral domain. Thus  $r = h E_{kj}$ . Therefore for each  $p \in e_j A e_i$ , we have  $f(p) = r p$  by (17). Consequently, there is a surjective  $Z(A)$ -module homomorphism

$$(18) \quad \begin{array}{ccc} e_k A e_j & \twoheadrightarrow & \mathrm{Hom}_{Z(A)}(e_j A e_i, e_k A e_i) \\ r & \mapsto & (p \mapsto r p). \end{array}$$

To show injectivity, suppose  $r, r' \in e_k A e_j$  are sent to the same homomorphism in  $\mathrm{Hom}_{Z(A)}(e_j A e_i, e_k A e_i)$ . Then for each  $p \in e_j A e_i$ ,

$$r p = r' p.$$

But  $e_j A e_i \neq 0$  by assumption (1). Whence  $r = r'$  since  $B$  is an integral domain. Therefore (18) is an isomorphism.

Similarly, there is a  $Z(A)$ -module isomorphism

$$\begin{array}{ccc} e_j A e_k & \xrightarrow{\sim} & \text{Hom}_{Z(A)}(e_i A e_j, e_i A e_k) \\ r & \mapsto & (p \mapsto pr). \end{array}$$

□

**Proposition 7.6.** *Suppose the arrows in  $Q_1^t$  are pair-wise coprime, and let  $a \in Q_1^t$ . Consider a simple matching  $D \in \mathcal{S}'$  for which  $x_D \mid \bar{a}$ . Set  $\mathbf{q} := \mathbf{q}_D$  and  $i := t(a)$ . Then for each  $j, k \in Q_0$ ,*

$$\text{Hom}_{\bar{R}}(e_j A_{\mathbf{q}} e_i, e_k A_{\mathbf{q}} e_i) \cong e_k A_{\mathbf{q}} e_j \quad \text{and} \quad \text{Hom}_{\bar{R}}(e_i A_{\mathbf{q}} e_j, e_i A_{\mathbf{q}} e_k) \cong e_j A_{\mathbf{q}} e_k.$$

*Proof.* Suppose the hypotheses hold. We claim that  $A_{\mathbf{q}}$  satisfies the assumptions of Lemma 7.5, with  $i = t(a)$  and arbitrary  $j, k \in Q_0$ .

Indeed, assumption (1) holds since there is a path between any two vertices of  $Q$ , and assumption (2) holds by Lemma 7.4.

To show that assumption (3) holds, fix  $j, k \in Q_0$ . Since  $\deg^+ i = 1$ ,  $x_D$  divides the  $\bar{\tau}_\psi$ -image of each nontrivial path in  $A e_i$ . By assumption (A), there is a path  $p \in e_j A e_i$  for which  $x_D^2 \nmid \bar{p}$  since  $D$  is a simple matching of  $A'$ . Fix a path  $q \in e_k A e_i$ .

We claim that there is a path  $r \in e_k A_{\mathbf{q}} e_j$  satisfying  $rp = q$ . Let  $p^+$  and  $q^+$  be lifts of  $p$  and  $q$  to  $Q^+$  with coincident tails,  $t(p^+) = t(q^+) \in Q_0^+$ . Let  $s \in e_k A e_j$  be a path for which  $s^+$  has no cyclic subpaths in  $Q^+$  and

$$t(s^+) = h(p^+) \quad \text{and} \quad h(s^+) = h(q^+).$$

Then by [B3, Lemma 2.3.2], there is some  $n \in \mathbb{Z}$  such that

$$\bar{s}p = \bar{q}\sigma^n.$$

First suppose  $n \leq 0$ . Set

$$r := \sigma_k^n s.$$

Then  $\bar{r}p = \bar{q}$ . Thus, since  $A$  is a homotopy algebra,  $rp = q$ .

So suppose  $n \geq 1$ ; without loss of generality we may assume  $n = 1$ . Then  $x_D^2 \mid \bar{s}p$  since  $x_D \mid \bar{q}$ . But  $x_D^2 \nmid \bar{p}$  by our choice of  $p$ . Thus

$$x_D \mid \bar{s}.$$

Therefore  $s$  factors into paths  $s = s_3 s_2 s_1$ , where  $s_2$  is a subpath of a unit cycle satisfying  $x_D \mid \bar{s}_2$ . Let  $b$  be one of the two paths for which  $b s_2$  is a unit cycle. Then  $x_D \nmid \bar{b}$  since  $x_D \mid \bar{s}_2$ . Thus  $b$  has vertex inverse

$$b^* \in e_{t(s_3)} A_{\mathbf{q}} e_{h(s_1)},$$

by Lemma 6.5. Set

$$r := s_3 b^* s_1.$$

Then  $\bar{r}s = \bar{q}$ . Therefore, again since  $A$  is a homotopy algebra,  $rp = q$ , proving our claim.  $\square$

**Theorem 7.7.** *Suppose the arrows in  $Q_1^t$  are pair-wise coprime. Let  $\mathfrak{q} \in \text{Spec } S$  be a minimal prime over  $\mathfrak{q} \cap R = \mathfrak{m}_0$ . Then there is some  $i \in Q_0$  for which*

$$A_{\mathfrak{q}} \cong \text{End}_{Z(A_{\mathfrak{q}})}(A_{\mathfrak{q}}e_i).$$

Furthermore,  $A_{\mathfrak{q}}e_i$  is a reflexive  $Z(A_{\mathfrak{q}})$ -module.

*Proof.* Suppose the hypotheses hold. By Theorem 5.7, there is some  $D \in \mathcal{S}'$  such that  $\mathfrak{q} = \mathfrak{q}_D$ . Since the arrows in  $Q_1^t$  are pair-wise coprime, there is a unique arrow  $a \in Q_1^t$  for which  $x_D \mid \bar{a}$ . Set

$$i := t(a) \quad \text{and} \quad \epsilon := \epsilon_D = 1_A - e_i.$$

For brevity, denote  $\text{Hom}_R(-, -)$  by  $R(-, -)$ . There are algebra isomorphisms

$$\begin{aligned} A_{\mathfrak{q}} &\cong \begin{bmatrix} e_i A_{\mathfrak{q}} e_i & e_i A_{\mathfrak{q}} \epsilon \\ \epsilon A_{\mathfrak{q}} e_i & \epsilon A_{\mathfrak{q}} \epsilon \end{bmatrix} \\ &\stackrel{(I)}{\cong} \begin{bmatrix} \tilde{R}(e_i A_{\mathfrak{q}} e_i, e_i A_{\mathfrak{q}} e_i) & \tilde{R}(\epsilon A_{\mathfrak{q}} e_i, e_i A_{\mathfrak{q}} e_i) \\ \tilde{R}(e_i A_{\mathfrak{q}} e_i, \epsilon A_{\mathfrak{q}} e_i) & \tilde{R}(\epsilon A_{\mathfrak{q}} e_i, \epsilon A_{\mathfrak{q}} e_i) \end{bmatrix} \\ &\stackrel{(II)}{\cong} \text{End}_{Z(A_{\mathfrak{q}})}(e_i A_{\mathfrak{q}} e_i \oplus \epsilon A_{\mathfrak{q}} e_i) \\ &= \text{End}_{Z(A_{\mathfrak{q}})}(A_{\mathfrak{q}}e_i), \end{aligned}$$

where (I) holds by Proposition 7.6 and (II) holds by Lemma 7.4.

Furthermore,  $A_{\mathfrak{q}}e_i$  is a reflexive  $Z(A_{\mathfrak{q}})$ -module:

$$\begin{aligned} {}_{Z(A_{\mathfrak{q}})}(Z(A_{\mathfrak{q}})(A_{\mathfrak{q}}e_i, Z(A_{\mathfrak{q}})), Z(A_{\mathfrak{q}})) &\stackrel{(I)}{=} ((A_{\mathfrak{q}}e_i, e_i A_{\mathfrak{q}} e_i), e_i A_{\mathfrak{q}} e_i) \\ &\stackrel{(II)}{=} (e_i A_{\mathfrak{q}}, e_i A_{\mathfrak{q}} e_i) \\ &\stackrel{(III)}{=} A_{\mathfrak{q}}e_i, \end{aligned}$$

where (I) holds by Lemma 7.4, and (II), (III) hold by Proposition 7.6.  $\square$

**Theorem 7.8.** *Let  $A$  be a nonnoetherian homotopy algebra satisfying assumptions (A) and (B). Then  $A_{\mathfrak{m}_0}$  is a nonnoetherian NCCR.*

*Proof.*  $A_{\mathfrak{m}_0}$  is nonnoetherian and an infinitely generated module over its nonnoetherian center by [B3, Lemma 4.55 and Theorem 4.65]; has a normal Gorenstein cycle algebra  $SR_{\mathfrak{m}_0}$  by Proposition 5.9; is cycle regular by Theorem 6.15; and for each prime  $\mathfrak{q} \in \text{Spec}(SR_{\mathfrak{m}_0})$  minimal over  $\mathfrak{m}_0$ , the cyclic localization  $A_{\mathfrak{q}}$  is an endomorphism ring of a reflexive  $Z(A_{\mathfrak{q}})$ -module by Theorem 7.7.  $\square$

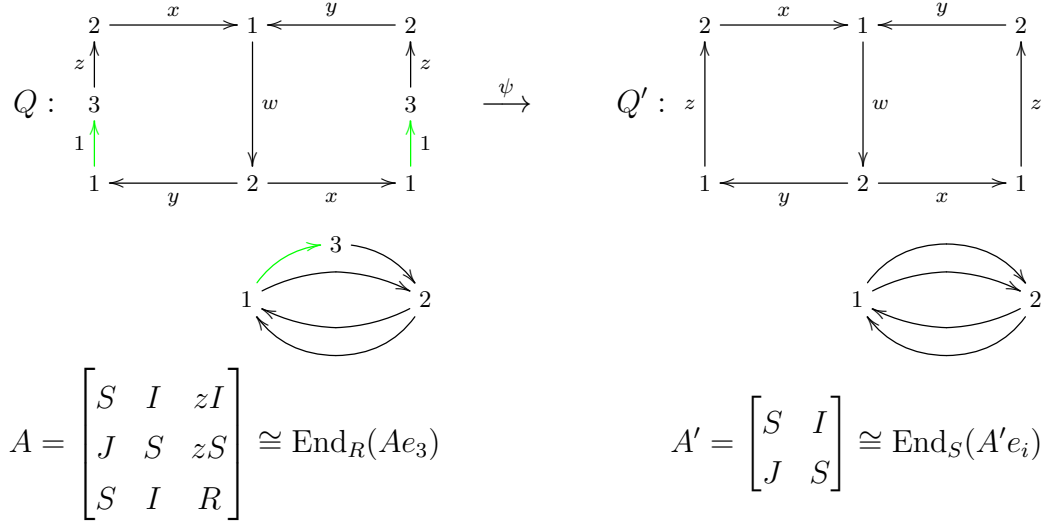


FIGURE 3. The homotopy algebra  $A$  is a nonnoetherian NCCR; see Section 7.1. The quivers  $Q$  and  $Q'$  on the top line are each drawn on a torus, and the contracted arrow of  $Q$  is drawn in green.

### 7.1. Examples.

**Example 7.9.** Set

$$B := k[x, y, z, w], \quad S := k[xz, yz, xw, yw] \cong k[a, b, c, d]/(ad - bc),$$

and

$$I := (x, y)S, \quad J := (z, w)S, \quad \mathfrak{m}_0 := zI, \quad R := k + \mathfrak{m}_0.$$

Consider the contraction of homotopy algebras given in Figure 3. Each arrow is labeled by its  $\bar{\tau}_\psi/\bar{\tau}$ -image in  $B$ . The center and cycle algebra of  $A$  are  $R$  and  $S$  respectively.

In this example, the maximal ideal  $\mathfrak{m}_0 \in \text{Max } R$  at the origin is a height one prime ideal of  $S$ .<sup>9</sup> Therefore  $\mathfrak{m}_0$  itself is the only one minimal prime of  $S$  over  $\mathfrak{m}_0$ . Furthermore, the cyclic localization of  $A$  at  $\mathfrak{m}_0$  is

$$A_{\mathfrak{m}_0} = \left\langle \begin{bmatrix} S_{\mathfrak{m}_0} & I & zI \\ J & S_{\mathfrak{m}_0} & zS \\ S & I & R_{\mathfrak{m}_0} \end{bmatrix} \right\rangle = \begin{bmatrix} S_{\mathfrak{m}_0} & IS_{\mathfrak{m}_0} & zIS_{\mathfrak{m}_0} \\ JS_{\mathfrak{m}_0} & S_{\mathfrak{m}_0} & zS_{\mathfrak{m}_0} \\ S_{\mathfrak{m}_0} & IS_{\mathfrak{m}_0} & R_{\mathfrak{m}_0} + \mathfrak{m}_0 S_{\mathfrak{m}_0} \end{bmatrix},$$

with center  $Z(A_{\mathfrak{m}_0}) \cong R_{\mathfrak{m}_0} + \mathfrak{m}_0 S_{\mathfrak{m}_0}$ .

<sup>9</sup>Note that the ideals  $xzS$  and  $yzS$ , each of which is properly contained in  $zI$ , are not prime since  $(xw) \cdot (yz) \in xzS$  and  $(xz) \cdot (yw) \in yzS$ .

**Example 7.10.** Set

$$B := k[x, y, z, w], \quad S := k[xz, yz, xw, yw],$$

and

$$I := (x, y)S, \quad J := (z, w)S, \quad \mathfrak{m}_0 := zwI^2, \quad R := k + \mathfrak{m}_0.$$

Consider the contraction of homotopy algebras given in Figure 1. As in Example 7.9, the center and cycle algebra of  $A$  are  $R$  and  $S$  respectively.

The minimal primes in  $S$  over  $\mathfrak{m}_0$  are

$$\mathfrak{q}_1 := zI \quad \text{and} \quad \mathfrak{q}_2 := wI,$$

each of height 1. The cyclic localizations of  $A$  at  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  are

$$A_{\mathfrak{q}_1} = \begin{bmatrix} S_{\mathfrak{q}_1} & IS_{\mathfrak{q}_1} & \mathfrak{q}_1 S_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} \\ wS_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} & zS_{\mathfrak{q}_1} & wS_{\mathfrak{q}_1} \\ S_{\mathfrak{q}_1} & IS_{\mathfrak{q}_1} & (k + \mathfrak{q}_1)_{\mathfrak{q}_1} + \mathfrak{q}_1 S_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} \\ S_{\mathfrak{q}_1} & IS_{\mathfrak{q}_1} & \mathfrak{q}_1 S_{\mathfrak{q}_1} & S_{\mathfrak{q}_1} \end{bmatrix} \cong \text{End}_{Z(A_{\mathfrak{q}_1})}(A_{\mathfrak{q}_1} e_3)$$

and

$$A_{\mathfrak{q}_2} = \begin{bmatrix} S_{\mathfrak{q}_2} & IS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & \mathfrak{q}_2 S_{\mathfrak{q}_2} \\ zS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & zS_{\mathfrak{q}_2} & wS_{\mathfrak{q}_2} \\ S_{\mathfrak{q}_2} & IS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & \mathfrak{q}_2 S_{\mathfrak{q}_2} \\ S_{\mathfrak{q}_2} & IS_{\mathfrak{q}_2} & S_{\mathfrak{q}_2} & (k + \mathfrak{q}_2)_{\mathfrak{q}_2} + \mathfrak{q}_2 S_{\mathfrak{q}_2} \end{bmatrix} \cong \text{End}_{Z(A_{\mathfrak{q}_2})}(A_{\mathfrak{q}_2} e_4),$$

with respective centers

$$Z(A_{\mathfrak{q}_1}) \cong (k + \mathfrak{q}_1)_{\mathfrak{q}_1} + \mathfrak{q}_1 S_{\mathfrak{q}_1} \quad \text{and} \quad Z(A_{\mathfrak{q}_2}) \cong (k + \mathfrak{q}_2)_{\mathfrak{q}_2} + \mathfrak{q}_2 S_{\mathfrak{q}_2}.$$

(Note that  $wS_{\mathfrak{q}_1} = JS_{\mathfrak{q}_1}$  since  $z = w \frac{xz}{xw}$ , and similarly  $zS_{\mathfrak{q}_2} = JS_{\mathfrak{q}_2}$ .) In contrast to Example 7.9,  $A$  itself is not an endomorphism ring, although its cyclic localizations are.

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